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Asymptotics of Diagonal Hermite–Padé Polynomials*

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Contents.

1. INTRODUCTION

The topic of this paper has its roots deep in the past in work on continued fractions, Padé approximants and orthogonal polynomials. As is well known, and summarized in Section 1.2, there is an intimate relation between orthogonal polynomials and the polynomials used in the definition of Padé approximants. This means that the classical asymptotic results of Bernstein and Szegö, described in Szegö's book [45], and reviewed in Section 1.1, imply corresponding results about the asymptotic behavior of the

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polynomials of high degree used in diagonal or near diagonal Padé approximants to functions of the appropriate form, Eq. (1.2.9).

Our purpose here is to present a conjecture, along with evidence in its support, which generalizes these results in two directions. In the first place we consider a broader class of functions having possibly complex branch points. Nowhere in the discussion will use be made of positivity or reality. Secondly, we treat the two types of Hermite–Padé polynomials (defined in Section 1.3) of which the polynomials used in Padé approximants are a special case.

It will be seen that the conjecture carries over two main features of the Bernstein–Szegö results. First of all, the leading part of the asymptotic behavior of the polynomials has a universal form for a large class of functions (corresponding to a class of weights in the classical case). In the second place, the more complete version of the asymptotic form is obtained by solving an appropriate Hilbert problem. In the case when m functions are being simultaneously approximated, the Hilbert problem is to be solved on a Riemann surface $\mathscr{R}$ of m sheets. The construction of $\mathscr{R}$ for a given set of functions is an important unsolved problem, although the case $m = 2$ is reasonably well understood (Section 3.4).

In Section 1.4, we outline how the conjecture applies to the case $m = 2$, Padé approximants, and discuss the general situation in Section 3. Section 4 contains a survey of some special cases where a rigorous discussion is possible, while Section 5 gives support on a heuristic or numerical basis. The conjecture has been constructed in order to be consistent with all the results of these two sections. In Section 6 we present some ideas on methods for proving the conjecture. The algebraic results of Section 2 could be omitted until they are required.

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1.1. *Bernstein–Szegö Orthogonal Polynomial Asymptotics*

One of the main results, stated as Theorem 12.1.2 by Szegö [45], applies to the polynomial $p(z)$ of degree n orthogonal with respect to the real weight $\omega(z)$ on the interval $L = \{z: -1 \leqslant z \leqslant 1\}$, so that

$$\int_L dz\, \omega(z)\, p(z)\, z^k = 0, \qquad k = 0, \ldots, n-1. \tag{1.1.1}$$

Szegö requires $\omega(z)$ to have the form

$$\omega(z) = (1 - z^2)^{-1/2}\, \sigma(z) \tag{1.1.2}$$

with $\sigma(z) \geqslant 0$, and the integrals

$$\int_0^\pi d\theta\, \sigma(\cos\theta), \qquad \int_0^\pi d\theta\, |\log \sigma(\cos\theta)| \tag{1.1.3}$$

must exist. In this case it is shown that $p(z)$ is unique up to normalization and Szegö's result is equivalent to

$$p(z) \underset{n\to\infty}{\sim} \chi(z) \tag{1.1.4}$$

for any complex z away from the line segment L. The function $\chi(z)$, analytic in the complex plane cut along L and proportional to z^n at ∞, may be characterized by the condition involving the limiting values of χ as $z \to L$ from opposite sides

$$\sigma(z)\, \chi_+(z)\, \chi_-(z) = 1, \qquad z \in L. \tag{1.1.5}$$

It may be shown that

$$\chi(z) = \exp(n\phi(z))\, h(z) \tag{1.1.6}$$

where $\exp(n\phi(z))$ satisfies (1.1.5) with $\sigma = 1$, and $h(z)$ is independent of n. Explicitly

$$\phi(z) = \log(z + (z^2 - 1)^{1/2}) \tag{1.1.7}$$

where the branch is chosen so that $\phi(z) \sim \log z$ near $z = \infty$. An explicit form for $h(z)$ is also readily available.

With further restrictions on the weight function it is possible to give the asymptotic form of $p(z)$ on L. Thus, Theorem 12.1.4 of [5] states that, provided $\sigma(z)$ is strictly positive, $z \in L$, and satisfies the smoothness condition

$$|\sigma(\cos(\theta + \delta)) - \sigma(\cos\theta)| < \text{const.}\, |\log \delta|^{-1-\lambda}, \qquad \lambda > 1 \tag{1.1.8}$$

then

$$p(z) = \chi_+(z) + \chi_-(z) + O[(\log n)^{-\lambda}]. \tag{1.1.9}$$

The zeros of $p(z)$ all lie on L and are distributed asymptotically with a density proportional to

$$|\phi'(z)| = |(z^2 - 1)^{-1/2}|. \tag{1.1.10}$$

All the above results are proved in Szegö's book [45], although in some cases we have changed the manner of presentation to fit in with our further development.

1.2. *Padé Approximants*

Suppose that $F(x)$ is analytic in x near to $x=0$. Polynomials $P_1(\rho, x)$, $P_2(\rho, x)$ of degrees ρ_1, ρ_2 respectively are defined by the condition

$$P_1(\rho, x) + P_2(\rho, x) F(x) = O(x^{(\rho_1+\rho_2+1)}). \tag{1.2.1}$$

The $[\rho_1/\rho_2]$ Padé approximant to $F(x)$ is

$$[\rho_1/\rho_2] = -P_1(\rho, x)/P_2(\rho, x). \tag{1.2.2}$$

Note that the polynomials $P_i(\rho, x)$ always exist but may not be unique.

There is a close connection between the denominator polynomial of a Padé approximant and an appropriate orthogonal polynomial. Thus take the case $\rho_1 = \rho_2 = n$ and write

$$p_i(z) = z^n P_i(\rho, x), \qquad i = 1, 2 \tag{1.2.3}$$

where $z = x^{-1}$. Then the polynomials $p_i(z)$ of degree n satisfy.

$$p_1(z) + f(z)\, p_2(z) = O(z^{-(n+1)}) \tag{1.2.4}$$

where $f(z) = F(z^{-1})$. If $p_2(z)$ is any solution of (1.2.4), then application of Cauchy's theorem to (1.2.4) shows that the corresponding $p_1(z)$ is given by

$$p_1(z) = -f(\infty)\, p_2(z) - \frac{1}{2\pi i} \int_\Gamma dt\, f(t)\, \frac{(p_2(t) - p_2(z))}{t - z} \tag{1.2.5}$$

where Γ is any contour enclosing all the singularities of $f(t)$ in a counter-clockwise sense.

For such a contour Γ (1.2.4) gives

$$\int_\Gamma dz\, f(z)\, p_2(z)\, z^k = 0, \qquad k = 0, \ldots, n-1. \tag{1.2.6}$$

Equation (1.2.6) continues to hold if Γ is distorted in any way that avoids crossing singularities of $f(z)$. In particular, if $f(z)$ is analytic outside L, then Γ may be collapsed around L and (1.2.6) becomes

$$\int_L dz (f_+(z) - f_-(z))\, p_2(z)\, z^k = 0, \qquad k = 0, \ldots, n-1 \tag{1.2.7}$$

provided that the integral makes sense. We see that we have obtained (1.1.1) if we set

$$\omega(z) = \text{const.}(f_+(z) - f_-(z)), \qquad z \in L, \tag{1.2.8}$$

and identify $p_2(z)$ with $p(z)$.

The conclusion is that any polynomial corresponding to the denominator of the $[n/n]$ Padé approximant to $F(x) = f(x^{-1})$ is an orthogonal polynomial in the sense (1.2.6) and the converse also holds. In particular a polynomial $p_2(z)$ corresponding to the function

$$f(z) = \int_L dt\, \frac{\omega(t)}{t - z} \tag{1.2.9}$$

is an orthogonal polynomial in the sense (1.1.1) and vice versa.

It should be noted that our use of the word "orthogonal" to describe polynomials satisfying (1.2.6) may differ from normal terminology. Brezinski [9] uses "general orthogonal polynomial" instead. We have called the polynomials discussed in Section 2.3 "generalized orthogonal polynomials."

From (1.2.5) we see that the error in the $[n/n]$ Padé approximant may be written

$$\begin{aligned} [n/n] - f(z) &= -p_1(z)/p_2(z) - f(z) \\ &= \frac{1}{2\pi i p_2(z)} \int_\Gamma dt\, \frac{f(t)\, p_2(t)}{t - z} \\ &= \frac{1}{2\pi i p_2^2(z)} \int_\Gamma dt\, \frac{f(t)\, p_2^2(t)}{t - z} \end{aligned} \tag{1.2.10}$$

using orthogonality. When $f(z)$ is of the form (1.2.9) we may use the results of Section 1.1 to see that the diagonal Padé approximants to $f(z)$ converge for any z outside L. This follows since, on L, $\operatorname{Re} \phi(z) = 0$, whereas elsewhere $\operatorname{Re} \phi(z) > 0$ and also the function $h(z)$, independent of n, does not vanish. Indeed the error satisfies the bound

$$|[n/n] - f(z)| < \text{const.} \exp(-2n\mu) \tag{1.2.11}$$

for any $\mu < \operatorname{Re} \phi(z)$.

1.3. *Hermite–Padé Polynomials*

The equation (1.2.1) used in the definition of Padé approximants may be generalized in two ways if several functions $F_j(x)$, $j = 1, \ldots, m$, analytic near $x = 0$ are available. For particular choices of $F_j(x)$ these forms were in fact given by Hermite [19, 20] before the notion of Padé approximants was introduced.

The polynomials of type I (Latin) $P_j(\rho, x)$, $j = 1,\ldots, m$, are defined by

$$\sum_{j=1}^{m} P_j(\rho, x)\, F_j(x) = O(x^{\sigma-1}) \tag{1.3.1}$$

with

$$\text{degree}\, P_j(\rho, x) \leqslant \rho_j - 1, \qquad j = 1,\ldots, m \tag{1.3.2}$$

and

$$\sigma = \sum_{j=1}^{m} \rho_j \tag{1.3.3}$$

for any set of positive integers ρ_j.

The polynomials of type II (German) $Q_j(\rho, x)$, $j = 1,\ldots, m$, satisfy

$$Q_i(\rho, x)\, F_j(x) - Q_j(\rho, x)\, F_i(x) = O(x^{\sigma+1}), \qquad i, j = 1,\ldots, m \tag{1.3.4}$$

with

$$\text{degree}\, Q_j(\rho, x) \leqslant \sigma - \rho_j, \qquad j = 1,\ldots, m. \tag{1.3.5}$$

As with Padé approximants, these polynomials always exist but may not be unique. Let be suppose throughout that $F_1(0) \neq 0$. The point $\rho = (\rho_1,\ldots, \rho_m)$ is called normal if every solution of (1.3.1) is such that the order of the right-hand side is exactly $\sigma - 1$. It follows [26] that, at a normal point, the solution of (1.3.1) is unique up to a constant multiplicative factor, and it may also be shown that the type II polynomials $\{Q_j(\rho, x)\}$ are unique up to a scalar multiple, with $Q_1(\rho, 0) \neq 0$.

In the case $m = 2$, both types of polynomials reduce to those involved in Padé approximants if we choose $F_1(x) = 1$.

It turns out, as explained in Section 2.1, that there are relations between the two types of polynomials that allow (in general) the determination of one set in terms of m sets of the other type.

1.4. *Outline of Conjecture—Padé Case*

It might be thought that the zeros of $p(z)$, the Bernstein–Szegö orthogonal polynomial, lie on L because this is the path used in the integration which defines orthogonality, but a little reflection shows this view to be incorrect. For suppose we take another arc $L' \in \mathbb{C}$ joining the points -1, 1. If $\omega(z)$ is analytic in that part of $\mathbb{C}$ lying between L and L' then (1.1.1) could also be written as

$$\int_{L'} dz\,\omega(z)\,p(z)\,z^k = 0, \qquad k = 0,..., n-1 \tag{1.4.1}$$

so that the same polynomial $p(z)$ results from orthogonality on L' and of course its zeros are on L, not on L'. We see that L appears to be preferred among arcs joining the points -1, 1 when we use our definition of orthogonality (The situation would be different if complex conjugation were involved. See Widom [47]).

Let us rephrase the same conclusion in the language of Padé approximants. The function $F(x) = f(z)$ corresponding to (1.2.9) represents a function with branch points at $x = \pm 1$. Padé approximants are calculated in terms of the values of $F(x)$ and its derivatives at $x = 0$ and these do not depend on how we might choose to insert a cut between the branch points, assuming again that $\omega(z)$ has enough analyticity to allow the possibility of different choices. In the example at hand we see that the Padé approximants, single-valued functions, converge to a function that is single valued in the plane cut along $z \in L$. The poles of the Padé approximants lie on this cut. It may be said that the Padé approximants have chosen this way of cutting the complex plane.

Part of the conjecture consists of the generalization [33] of these results of orthogonal polynomials/Padé approximants corresponding to functions $f(z)$ with a form more general than (1.4.1). We believe that for any appropriate $f(z)$ there will be a preferred set of arcs in the complex plane S, with connected complement, which is completely determined in terms of some of the branch points of $f(z)$. As $n \to \infty$, all but a bounded number of zeros of $p(z)$ will approach S and correspondingly $[n/n]$ will converge in capacity to $f(z)$ for $z \notin S$.

Of course, for a given function $f(z)$ there has to be a unique S, but the problem of determining such an S will be laid aside (until Section 3.4) and instead we will follow the approach of Szegö and consider functions $f(z)$ constructed from a particular preferred set S. Thus we choose a particular S and suppose that $f(z)$ is analytic in the complex plane cut along S, so that we may write

$$f(z) = \int_S dt\,\frac{\omega(t)}{t-z}. \tag{1.4.2}$$

The conjecture describes the asymptotic form of polynomials $p(z)$ associated with this $f(z)$, provided the weight function $\omega(z)$ satisfies certain conditions. The principal condition is that $\omega(t)$, which may be complex, should not have more than a finite number of zeros, $z \in S$. In addition it should be adequately smooth, as required in the Bernstein–Szegö theory.

The function $\phi(z)$, defined generally in Section 3.1, plays an important role in the discussion. In the present case, $\phi(z)$ may be defined as the complex Green's function with pole at infinity for the complement of S, so that $\phi(z)$ is analytic, $z \in \mathbb{C} - S$, except near $z = \infty$, where $\phi(z) \sim \log z + \text{const.}$, and $\operatorname{Re} \phi(z) = 0$, $z \in S$. The conjecture implies that the dominant part of the asymptotic behavior of $p(z)$ is $\exp(n\phi(z))$, except near S.

A more precise statement of the asymptotic conjecture for $p(z)$ $(=p_2(z)$ of Section 1.2) is

$$p(z) \underset{n \to \infty}{\sim} \chi(z), \qquad z \notin S \tag{1.4.3}$$

except near zeros of $\chi(z)$. The function $\chi(z)$ is analytic, $z \in \mathbb{C} - S$, except for a pole of order n at ∞. In conjunction with the function $R(z)$, analytic, $z \in \mathbb{C} - S$, with a zero of order $(n+1)$ at ∞, $\chi(z)$ satisfies the condition (equivalent to (3.2.11))

$$\begin{aligned} \omega(z)\,\chi_+(z) &= R_-(z) \\ \omega(z)\,\chi_-(z) &= R_+(z) \end{aligned} \qquad z \in S. \tag{1.4.4}$$

This is equivalent to a Hilbert problem on the two-sheeted Riemann surface formed by joining two copies of $\mathbb{C} - S$ at S. Its solution is given in (4.3.32), where we use (4.3.33) to relate $\rho(z)$ to $\omega(z)$. There is a form analogous to (1.1.9) for the behavior of $p(z)$, $z \in S$, and corresponding predictions for $p_1(z)$.

In the case when $S = L$ and $\omega(z)$ is real, positive, it is easy to see that the above formulae are equivalent to the results quoted in Section 1.1.

2. Algebraic Preliminaries

In this section we describe a number of algebraic properties of Hermite–Padé polynomials that will be useful in the subsequent discussion. The section could be omitted on first reading and referred to later as required.

2.1. *Relation between H–P Polynomials of Different Types*

The H–P polynomials $P_i(\rho, x)$, $Q_i(\rho, x)$ were introduced in Section 1.3. Let us choose

$$\rho_i^{(k)} = \eta_i + \delta_{ik}, \qquad i, k = 1, \ldots, m \tag{2.1.1}$$

and

$$\mu_i^{(k)} = \eta_i - \delta_{ik}, \qquad i, k = 1, \ldots, m \tag{2.1.2}$$

where η_i, $i = 1,\dots, m$, is a set of positive integers. It follows from the work of Mahler [28], Jager [23] and Loxton and Van der Poorten [26] that if the points $\rho^{(k)}$, $\mu^{(k)}$, $k = 1,\dots, m$, as well as the point $(\eta_1,\dots, \eta_m)$ are normal, then

$$P^T Q = x^\eta I \tag{2.1.3}$$

where the matrices P, Q are defined by

$$P_{ij} = P_i(\rho^{(j)}, x), \qquad Q_{ij} = Q_i(\mu^{(j)}, x). \tag{2.1.4}$$

and

$$\eta = \sum_{i=1}^{m} \eta_i. \tag{2.1.5}$$

An analogous result was known to Van Vleck [46] and earlier authors.

To begin with it may be shown [23, 28] that, under the above normality conditions, normalizations may be chosen so that

$$\begin{aligned} \det P &= x^\eta \\ \det Q &= x^{(m-1)\eta}. \end{aligned} \tag{2.1.6}$$

Now, with $F_1(0) \neq 0$, we have

$$\begin{aligned} F_1(x) \sum_{i=1}^{m} P_i(\rho^{(j)}, x)\, Q_i(\mu^{(k)}, x) &= \sum_{i=1}^{m} P_i(\rho^{(j)}, x)\, F_i(x)\, Q_1(\mu^{(k)}, x) + O(x^\eta) \\ &= Q_1(\mu^{(k)}, x)\, O(x^\eta) + O(x^\eta) \end{aligned} \tag{2.1.7}$$

from (1.3.4) and then (1.3.1). Thus

$$\sum_{i=1}^{m} P_i(\rho^{(j)}, x)\, Q_i(\mu^{(k)}, x) = O(x^\eta). \tag{2.1.8}$$

The degree of the left-hand side is at most $\eta - 1, j \neq k$ and $\eta, j = k$. Thus for $j \neq k$ (2.1.8) gives zero and for $j = k$, const. x^η. None of these constants can be zero for then $\det[P^T Q] = 0$ in contradiction with (2.1.6). A suitable choice of normalization gives (2.1.3).

Note that (2.1.3) implies

$$Q P^T = x^\eta I \tag{2.1.9}$$

so that in particular

$$\sum_{l=1}^{m} P_j(\rho^{(l)}, x)\, Q_1(\mu^{(l)}, x) = 0, \qquad j = 2,\dots, m, \tag{2.1.10}$$

a formula that is important in our construction of reproducing kernels.

2.2. *Generalized Orthogonal Polynomials*

In analogy with the connection between Padé polynomials and orthogonal polynomials described in Section 1.2, we give relations between H–P polynomials and certain generalized orthogonal polynomials. This connection was first discovered by Angelesco [2].

For this section and the next we suppose that $F_1(x) = 1$ and choose $\eta_j = n+1, j = 1,..., m$, for use in the polynomials of Section 2.1.

Let us consider polynomials $p_k(z)$, $k = 2,..., m$ of degree n defined by

$$p_k(z) = z^n P_k(\rho^{(1)}, z^{-1}), \qquad k = 2,..., m. \tag{2.2.1}$$

Multiplying (1.3.1) by z^{n+j+1} and integrating round a closed contour large enough to include all the singularities of $F_k(z^{-1})$ leads, with $\nu = (m-1)(n+1)$, to

$$\sum_{k=2}^{m} \int dz\, p_k(z)\, zF_k(z^{-1})\, z^j = 0, \qquad j = 0,..., \nu - 2. \tag{2.2.2}$$

Equation (2.2.2) could be regarded as a generalized orthogonality relation for the "vector" polynomial $\{p_k(z)\}$ with respect to the "vector" weight $\{zF_k(z^{-1})\}$.

Polynomials roughly dual to $p_k(z)$ may be constructed as

$$q^{(l)}(z) = z^{\nu-1} Q_1(\mu^{(l)}, z^{-1}), \qquad l = 2,..., m. \tag{2.2.3}$$

The degree of $q^{(l)}(z)$ is $\nu - 1$ and it satisfies the orthogonality relations

$$\begin{aligned} &\int dz\, q^{(l)}(z)\, zF_l(z^{-1})\, z^j = 0, \qquad j = 0,..., n-1 \\ &\int dz\, q^{(l)}(z)\, F_k(z^{-1})\, z^j = 0, \qquad j = 0,..., n;\ k = 2,..., m,\ k \neq l. \end{aligned} \tag{2.2.4}$$

In (2.2.4) and Section 2.3 the integration contour is as above.

The two types of H–P polynomials correspond to the two types of generalized orthogonal polynomial that are naturally defined when several weight functions are available. These polynomials should be distinguished from those that can be defined in terms of a square matrix weight function, a case not considered in this article.

The relation between Padé approximants and orthogonal polynomials has led to a better understanding of the former, and we expect this situation to be repeated for H–P polynomials.

2.3. *Reproducing Kernels*

A kernel that has the property of reproducing the operand when applied to polynomials of degree n played a key role in the proof of the results of Section 1.1. The polynomials in question were orthogonal on the unit circle but the same method was used by Nuttall and Singh [31] to study certain polynomials orthogonal in the sense of Section 1.2. It is likely that an appropriate reproducing kernel will be important in proving asymptotic results for H–P polynomials. More details on this possibility will be found in Section 6.1.

Now suppose that we choose $F_1(x)=1$, and take $\eta_j=n+1, j=1,..., m$, for use in the polynomials of Section 2.1. A family of reproducing kernels $K_{jk}(z,t)$ may be defined as

$$K_{jk}(z,t)=z^{n+1}\sum_{l=1}^{m} P_j(\rho^{(l)},z^{-1})\,Q_1(\mu^{(l)},t^{-1})\,t^{\nu}F_k(t^{-1})(t-z)^{-1}, \qquad j,k=2,..., m \quad (2.3.1)$$

for each of which

$$\int dt\,K_{jk}(z,t)\,\pi(t)=2\pi i\,\delta_{jk}\pi(z) \tag{2.3.2}$$

for any polynomial π of degree n.

The proof of (2.3.2) is analogous to that given by Szegö [45]. It stems from relation (2.1.10). We write

$$\pi(t)=(\pi(t)-\pi(z))+\pi(z) \tag{2.3.3}$$

and note that $(\pi(t)-\pi(z))(t-z)^{-1}$ is a polynomial of degree $n-1$ and so the corresponding part of (2.3.3) contributes zero to (2.3.2) because of the orthogonality relation (2.2.4), and a corresponding relation for $l=1$. We are left with

$$\begin{aligned}\int dt\,K_{jk}(z,t)\,\pi(t)&=\pi(z)\int dt\,K_{jk}(z,t)\\&=\pi(z)\sum_{l=1}^{m}\int dt\,Q_1(\mu^{(l)},t^{-1})\,t^{\nu}F_k(t^{-1})\\&\quad\times[P_j(\rho^{(l)},z^{-1})\,z^{n+1}-P_j(\rho^{(l)},t^{-1})\,t^{n+1}](t-z)^{-1}\end{aligned} \tag{2.3.4}$$

using (2.1.10). Now the expression $[\;](t-z)^{-1}$ is a polynomial in t of degree n for which the coefficient of t^n is $-P_j(\rho^{(l)},0)$. The lower powers of t again contribute nothing to the integral because of (2.2.4) so that

$$\int dt\,K_{jk}(z,t)\,\pi(t)=\lambda_{jk}\pi(z) \tag{2.3.5}$$

where

$$\lambda_{jk} = -\sum_{l=1}^{m} P_j(\rho^{(l)}, 0) \int dt\, Q_1(\mu^{(l)}, t^{-1})\, t^{\nu+n} F_k(t^{-1}). \tag{2.3.6}$$

Using (2.1.10) and then (1.3.4) we have

$$\begin{aligned} \lambda_{jk} &= \sum_{l=1}^{m} \int dt\, Q_1(\mu^{(l)}, t^{-1})\, F_k(t^{-1})[P_j(\rho^{(l)}, t^{-1}) - P_j(\rho^{(l)}, 0)]\, t^{\nu+n} \\ &= \sum_{l=1}^{m} \int dt \{Q_k(\mu^{(l)}, t^{-1}) + O(t^{-\nu-n-1})\}[P_j(\rho^{(l)}, t^{-1}) - P_j(\rho^{(l)}, 0)]\, t^{\nu+n} \end{aligned} \tag{2.3.7}$$

The term containing $O(t^{-\nu-n-1})$ gives zero on counting powers, and, of the remainder, the term with $P_j(\rho^{(l)}, 0)$ is zero by Cauchy's theorem. We are left with

$$\lambda_{jk} = \sum_{l=1}^{m} \int dt\, Q_k(\mu^{(l)}, t^{-1})\, P_j(\rho^{(l)}, t^{-1})\, t^{\nu+n} \tag{2.3.8}$$

which from (2.1.9)

$$= \delta_{jk} \int dt\, t^{-1}$$

$$= 2\pi i \delta_{jk}.$$

2.4. *Multiple Integral Formula for Generalized Orthogonal Polynomials*

Suppose that we are given weight functions $\omega_i(z)$, $i = 2,..., m$, defined on a contour Γ in the complex z-plane. We define generalized orthogonal polynomials of type *I*, $p_j(z)$, $j = 2,..., m$, of degree μ_j, respectively, by the conditions

$$\sum_{j=2}^{m} \int dz\, \omega_j(z)\, p_j(z)\, z^k = 0, \qquad k = 0,..., (\mu_2 + \cdots + \mu_m + m - 3). \tag{2.4.1}$$

Such polynomials exist but may not be unique.

In generalization of the result quoted by Szegö [45] there exists an explicit representation of $p_j(z)$ in the form of a multiple integral (10). It is

$$\begin{aligned} p_j(z) &= (\mu_j + 1)(-1)^{\mu_2 + \cdots + \mu_{j-1}} \\ &\times \left\{ \prod_{\substack{l=2 \\ l \neq j}}^{m} \left[\prod_{k=0}^{\mu_l} \left(\int_\Gamma dz_k^{(l)} \omega_l(z_k^{(l)}) \right) \right] V(z_0^{(l)} \cdots z_{\mu_l}^{(l)}) \right\} \\ &\times \left\{ \prod_{k=1}^{\mu_j} \left(\int_\Gamma dz_k^{(j)} \omega_j(z_k^{(j)}) \right) \right\} V(z z_1^{(j)} \cdots z_{\mu_j}^{(j)}) \\ &\times V(z_0^{(2)} \cdots z_{\mu_2}^{(2)} z_0^{(3)} \cdots z_{\mu_3}^{(3)} \cdots z_1^{(j)} \cdots z_{\mu_j}^{(j)} \cdots z_0^{(m)} \cdots z_{\mu_m}^{(m)}) \end{aligned} \tag{2.4.2}$$

where the Vandermonde determinant $V(x_1 \cdots x_s)$ is given by

$$V(x_1 \cdots x_s) = \det[x_i^{j-1}], \qquad i, j = 1,..., s. \tag{2.4.3}$$

It is possible that (2.4.2) is identically zero, but if this is not the case (2.4.2) furnishes a set of polynomials satisfying (2.4.1). If the set of polynomials for the given set of degrees is unique up to a constant factor, then (2.4.2) provides a representation.

The generalized orthogonal polynomial of type II, $q(z)$, of degree $\mu_2 + \cdots + \mu_m + m - 1$ is defined by

$$\int_\Gamma dz\, \omega_j(z)\, q(z)\, z^k = 0, \qquad k = 0,..., \mu_j; j = 2,..., m. \tag{2.4.4}$$

We find

$$q(z) = \left\{ \prod_{l=2}^{m} \left[\prod_{k=0}^{\mu_l} \left(\int_\Gamma dz_k^{(l)} \omega_l(z_k^{(l)}) \right) \right] V(z_0^{(l)} \cdots z_{\mu_l}^{(l)}) \right\}$$
$$\times V(z z_0^{(2)} \cdots z_{\mu_2}^{(2)} \cdots z_0^{(m)} \cdots z_{\mu_m}^{(m)})$$

The proof proceeds along lines similar to those of [10] for (2.4.2).

2.5. *Christoffel's Formula*

Christoffel's formula [45] tells us how to construct an orthogonal polynomial corresponding to weight $\theta(z)\,\omega(z)$ in terms of polynomials orthogonal with respect to weight $\omega(z)$, where

$$\theta(z) = \prod_{j=1}^{l} (z - z_l). \tag{2.5.1}$$

If the former polynomial of degree n is denoted by $q(z)$ and the latter by $p(n, z)$, we have, provided the determinant is not identically zero,

$$\theta(z)\, q(z) = \begin{vmatrix} p(n, z) & p(n+1, z) & \cdots & p(n+l, z) \\ p(n, z_1) & p(n+1, z_1) & \cdots & p(n+l, z_1) \\ \vdots & \vdots & & \vdots \\ p(n, z_l) & p(n+1, z_l) & \cdots & p(n+l, z_l) \end{vmatrix}. \tag{2.5.2}$$

This result may be generalized to the case of H–P polynomials. We give one example. Suppose $m = 3$ and we are given type I H–P polynomials $\{P_i(\rho, x)\}$ corresponding to functions F_1, F_2, F_3. We wish to construct polynomials $\{\bar{P}_i(\rho, x)\}$ corresponding to functions $\theta_j(x)\, F_j, j = 1, 2, 3$, where

$$\theta_j(x) = \prod_{k=1}^{l_j} (x - x_k^{(j)}), \qquad j = 1, 2, 3. \tag{2.5.3}$$

We set

$$\theta_j(x)\,\bar{P}_j(\rho, x) = \sum_{\alpha} a_\alpha P_j(\rho + \alpha, x), \qquad j = 1, 2, 3, \tag{2.5.4}$$

where the lattice points α are

$$\alpha = (000)(100) \cdots (l_1 00)(010) \cdots (0 l_2 0)(001) \cdots (00 l_3). \tag{2.5.5}$$

It is seen that $\{\bar{P}_j\}$ will satisfy the appropriate equation of the form (1.3.1) and will be a polynomial of correct degree provided that the right-hand side of (2.5.4) vanishes at the zeros of θ_j. Thus we have

$$\sum_{\alpha} a_\alpha P_j(\rho + \alpha, x_k^{(j)}) = 0, \qquad k = 1,..., l_j;\, j = 1, 2, 3. \tag{2.5.6}$$

Equation (2.5.6) constitutes equations for the coefficients a_α that in general may be solved to give a determinantal form for $\{\bar{P}_j\}$ analogous to (2.5.2).

3. Conjecture on Asymptotic Behavior

The conjecture on the asymptotic behavior of Hermite–Padé polynomials to a set of functions $F_j(x)$, $j = 1,..., m$, is stated in terms of an appropriate Riemann surface $\mathscr{R}$. Rather as for the Padé case, $m = 2$, discussed in Section 1.4, and the special case of this, Sections 1.1, 1.2, we first choose a Riemann surface $\mathscr{R}$ with m sheets and then construct sets of functions for which $\mathscr{R}$ will be the appropriate surface, in analogy with the choice (1.4.2). In Section 3.2 we treat one class of such sets for which the conjecture is clearest and the analysis of its predictions easiest. An extended class of sets of functions is discussed in Section 3.3, but a complete formulation of the conjecture is still lacking.

Given functions $\{F_j(x)\}$, there should be a most one surface $\mathscr{R}$ for which the functions meet the conditions of the conjecture. Only in the case $m = 2$ have we made any progress in determining the appropriate $\mathscr{R}$, and these ideas are outlined in Section 3.4.

3.1. *Riemann Surface*

In analogy with what might be called the Bernstein–Szegö point of view we begin with a Riemann surface $\mathscr{R}$ with m sheets, each a copy of the complex z-plane. The surface $\mathscr{R}$ may be described by an equation of the form

$$r(y, z) = 0 \tag{3.1.1}$$

where $r(y, z)$ is an irreducible polynomial in y, z of degree m in y.

Below we shall specify how points on $\mathscr{R}$ are to be assigned to the various sheets. We shall use the notation $z^{(k)}$ to indicate the point on sheet k of $\mathscr{R}$ above the point z. From time to time, where the context makes the meaning clear, we shall use z to denote a point on $\mathscr{R}$ or a point in $\mathbb{C}$.

The assignment is made with the help of a function ϕ defined on $\mathscr{R}$. It is an Abelian integral of the third kind (42) with poles at $\infty^{(j)}, j = 1,..., m$. At $z = \infty^{(1)}$, which we take to correspond to the $x = 0$ about which we are expanding in (1.3.1), the residue of ϕ is $(m-1)$ and at $\infty^{(j)}, j = 2,..., m$, the residue is -1. This means that

$$\begin{aligned} &\phi(z) \sim -(m-1) \log z, && z \sim \infty^{(1)} \\ &\phi(z) \sim \log z, && z \sim \infty^{(j)}, \quad j = 2,..., m. \end{aligned} \tag{3.1.2}$$

Elsewhere ϕ is analytic in the local variable. The function ϕ is not single-valued on $\mathscr{R}$ but around any cycle on $\mathscr{R}$ (cycles exist which are not homotopic to zero if the genus g of $\mathscr{R}$ is greater than zero) the change in ϕ is pure imaginary. According to Siegel [42] such a function is unique up to an additive constant and $\operatorname{Re} \phi$ is single valued on $\mathscr{R}$.

We use the value of $\operatorname{Re} \phi$ to prescribe the m sheets of $\mathscr{R}$. The sheets are labelled so that, for each $z \in \mathbb{C}$,

$$\operatorname{Re} \phi(z^{(m)}) \geqslant \operatorname{Re} \phi(z^{(m-1)}) \geqslant \cdots \geqslant \operatorname{Re} \phi(z^{(1)}). \tag{3.1.3}$$

We denote by $s \in \mathscr{R}$ the boundary between sheets m, $m-1$. This consists of one or more non-intersecting closed curves s_i, $i = 1,....$ The boundary s separates $\mathscr{R}$ into two open sets $\mathscr{R}_m = \{z^{(m)}: \operatorname{Re} \phi(z^{(m)}) > \operatorname{Re} \phi(z^{(m-1)})\}$ and $\mathscr{R}_0 = \mathscr{R} - \mathscr{R}_m - s$. We shall assume in Section 3.2, 3.3 that $\mathscr{R}_0$, the first $(m-1)$ sheets of $\mathscr{R}$, consists of a single connected (perhaps multiply connected) component. Perhaps this is always the case, for we know of no counter examples. Sheet m, $\mathscr{R}_m$, may have several disconnected components. We denote by $+$ that side of s bounding $\mathscr{R}_0$, $-$ the other side, and orient s so that the $+$ side is on the left.

The image of s in the complex plane we call S, so that

$$S = \{z \in \mathbb{C}: \operatorname{Re} \phi(z^{(m)}) = \operatorname{Re} \phi(z^{(m-1)})\} \tag{3.1.4}$$

S consists of a set of analytic arcs, and the mapping $s \to S$ is generally $2 \to 1$.

In a similar way we introduce $s' \in \mathscr{R}$ as the boundary between sheets 1, 2 of $\mathscr{R}$ and let $S' \in \mathbb{C}$ be the image of s' in the complex plane, so that

$$S' = \{z \in \mathbb{C}: \operatorname{Re} \phi(z^{(1)}) = \operatorname{Re} \phi(z^{(2)})\}. \tag{3.1.5}$$

Examples of the function $\phi(z)$ and the sets S, S' are discussed for various surfaces $\mathscr{R}$ in Sections 4, 5.

We now take the opportunity to remind the reader of a few facts about functions on $\mathscr{R}$ that will subsequently be needed often. An excellent reference is Siegel [42].

There are g linearly independent Abelian differentials of the first kind, taken to be dw_k, $k=1,\dots,g$, which are integrable over every path on $\mathscr{R}$. Each differential has $2g$ independent period integrals Ω_{kj}, $k=1,\dots,g$; $j=1,\dots,2g$, obtained by integrating over each of $2g$ cycles on $\mathscr{R}$ that form a basis for the fundamental group.

Meromorphic functions on $\mathscr{R}$ are rational functions of y, z. They are meromorphic in the local variable at each point of $\mathscr{R}$. It may be shown that every meromorphic function has the same number of zeros as poles. Unless $g=0$, it is not possible to construct a meromorphic function with an arbitrary set of zeros and poles. Indeed, if $\alpha_1,\dots,\alpha_\nu$ and $\beta_1,\dots,\beta_\nu$ are the poles, zeros of a meromorphic function, Abel's theorem states that there exist integers n_i such that

$$\sum_{j=1}^{\nu} \int_{\alpha_j}^{\beta_j} dw_k = \sum_{i=1}^{2g} n_i \Omega_{ki}, \qquad k=1,\dots,g \tag{3.1.6}$$

and conversely.

Suppose that all poles, zeros, but $\beta_1,\dots,\beta_g$ are chosen. Then (3.1.6) leads to an example of the Jacobi inversion problem in which $\beta_1,\dots,\beta_g$ and integers $\{n_i\}$ are to be found from g equations of the form

$$\sum_{j=1}^{g} \int_{\alpha_j}^{\beta_j} dw_k = W_k + \sum_{i=1}^{2g} n_i \Omega_{k_i}, \qquad k=1,\dots,g \tag{3.1.7}$$

with W_k, $k=1,\dots,g$, given. The Jacobi inversion problem always has a solution for the set of points $\beta=\beta_1,\dots,\beta_g$ (called a divisor). In general, the solution is unique but it is possible, if $g>1$, that a solution β may be a special divisor, in which case the solution is not unique.

To define a special divisor, assuming that $\beta_1,\dots,\beta_g$ are all distinct, we form the $g\times g$ matrix with elements $dw_k/d\beta_j$ using the local parameter at each point. If the rank ρ of this matrix is less than g, then β is special of rank ρ. There is a corresponding definition in the case when $\beta_1,\dots,\beta_g$ are not all distinct.

If one solution β of (3.1.7) is a special divisor of rank ρ, then all solutions are special of rank ρ. Moreover, the general solution is obtained by choosing points $\beta_1,\dots,\beta_{g-\rho}$ arbitrarily, in which case the remaining points of β are determined. The situation is of course exactly the same for the problem of determining zeros $\beta_1,\dots,\beta_g$ of a meromorphic function with its other zeros and poles given.

3.2. *Conjecture—Case* 1

For what we call case 1 of the conjecture, functions $F_j(x) = f_j(z)$, $j = 1,..., m$, must exist, which obey the following conditions.

(i) Each function $f_j(z)$ is analytic and single-valued, $z \in \mathscr{R}_0$, which is connected.

(ii) Except perhaps for a finite number of points, each function $f_j(z)$ must have a limit $f_j(z+)$ as $z \to s$ and the limit must be adequately smooth on s.

(iii) Suppose that $z^{(m)} \in s$. There will be another point $z^{(m-1)} \in s$ corresponding to the same $z \in \mathbb{C}$. We define

$$D(z^{(m)}) = \det(f_i(z^{(j)})), \qquad i, j = 1,..., m,\ z^{(m)} \in s \tag{3.2.1}$$

where, for $j = m-1, m$, $f_i(z^{(j)}+)$ is implied. Then we require that $D(z^{(m)}) \neq 0$ except for at most a bounded number of points $z^{(m)} \in s$.

We remark that analytic means analytic in the local variable on $\mathscr{R}$, so that, in terms of $z \in \mathbb{C}$, $\{f_j(z)\}$ might have branch points on $\mathscr{R}_0$. Analytic could probably be changed to meromorphic with little difficulty.

In general, $\mathscr{R}_0$ will not be simply connected, in which case the condition of single valuedness means that $f_i(z)$ must not change when taken round any cycle contained in $\mathscr{R}_0$. It could turn out that $\mathscr{R}_0$ is simply connected, in which case analyticity ensures single valuedness. We call this case 1a.

We begin with type I polynomials. The conjecture is less complicated if we assume that the degree $(\rho_j - 1)$ of each polynomial is the same $\rho_j - 1 = n$, $j = 1,..., m$. We set

$$P_j(\rho, x) = x^n p_j(z), \qquad j = 1,..., m \tag{3.2.2}$$

so that each $p_j(z)$ is a polynomial in z of degree n satisfying

$$\sum_{j=1}^{m} p_j(z) f_j(z^{(1)}) = O(z^{-(m-1)(n+1)}), \qquad z \to \infty. \tag{3.2.3}$$

The conjecture asserts that the asymptotic form of $p_j(z)$ as $n \to \infty$ is given by the solution of a boundary value problem on $\mathscr{R}$. There are functions $\chi_j(z)$, $j = 1,..., m$, which are meromorphic for $z \in \mathscr{R}_m$ and a meromorphic function $R(z)$, $z \in \mathscr{R}_0$ which is an approximation to $\sum_{j=1}^{m} p_j(z) f_j(z)$. These functions obey

$$\sum_{j=1}^{m} f_j(z^{(k)}) \chi_j(z^{(m)}) = 0, \qquad k = 1,..., m-1,\ z \notin S \tag{3.2.4}$$

$$\sum_{j=1}^{m} f_j(z+) \chi_j(z-) = R(z+), \qquad z \in s. \tag{3.2.5}$$

We note that there is a 1–1 correspondence between points $z \in \mathscr{R}_m$ and $z \in \mathbb{C}$ with $z \notin S$, and sometimes we shall think of $\chi_j(z)$ as a function of $z \in \mathbb{C}$, $z \notin S$.

We predict that, except near to zeros of $\chi_j(z)$, $z \notin S$,

$$p_j(z) \underset{n\to\infty}{\sim} \chi_j(z), \qquad z \notin S, j = 1,..., m. \tag{3.2.6}$$

The polynomial $p_j(z)$ will have a zero near to each zero of $\chi_j(z)$, $z \notin S$. In addition we predict, in analogy to (1.1.9), that, for points $z \in S$, where the functions $f_j(z+)(z \in s)$ are sufficiently smooth,

$$p_j(z) \underset{n\to\infty}{\sim} \chi_{j+}(z) + \chi_{j-}(z), \qquad z \in S \tag{3.2.7}$$

where the subscript + means the limit from the left side of S, and a proviso about zeros similar to above is implied.

To complete the conjecture it is necessary to give information on the poles, zeros, and boundary behavior of $\{\chi_j(z)\}$, $R(z)$, sufficient to make the solution of (3.2.4), (3.2.5) unique (up to a constant factor; in future we imply this qualification), except in cases when $\{p_j(z)\}$ are not unique or almost so. The reader will observe that the $\{\chi_j(z)\}$, $R(z)$, of any solution of (3.2.4), (3.2.5) may be multiplied by a function meromorphic on $\mathscr{R}$ to obtain another solution of these equations. Because of ignorance of the correct specification of the boundary behavior on s, we do not present here a general statement of the conditions required for uniqueness, but the problem is solved for a number of examples in Section 4. Common to all cases, however, supposing no point at ∞ lies on s, is the requirement that each $\chi_j(z)$ has a pole of order n at $\infty^{(m)}$, and that $R(z)$ has poles of order n at $\infty^{(k)}$, $k = 2,..., m-1$, and a zero of order $(m-1)(n+1)$ at $\infty^{(1)}$.

The conjecture implies that the dominant factor in each $p_j(z)$ for large n is $\exp(n\phi(z^{(m)}))$. All the zeros of $p_j(z)$ except at most a number independent of n will lie near to S for large n. Asymptotically their line density will be proportional to $|\phi'(z^{(m)}) - \phi'(z^{(m-1)})|$. Some or all of the remaining zeros will vary in position with n, and the nature of this variation will become apparent as the examples are studied.

The problem (3.2.4), (3.2.5) may be reduced to one of more familiar form. For $z^{(m)} \in \mathscr{R}_m$ we introduce the cofactors $A_i(z^{(m)})$, $i = 1,..., m$, of the element $f_i(z^{(m)})$ in the determinant of (3.2.1). Of course, with $z^{(m)} \in \mathscr{R}_m$, $z^{(m-1)} \in \mathscr{R}_0$, and we see that each $A_i(z^{(m)})$ is piecewise analytic, $z^{(m)} \in \mathscr{R}_m$. Each cofactor changes sign as we cross a curve in $\mathscr{R}_m$ that is the image of a boundary between two adjacent sheets of $\mathscr{R}$. If we introduce $g_j(z)$, $j = 1,..., m$, a set of independent functions meromorphic on $\mathscr{R}$, and define

$$G(z^{(m)}) = \det(g_i(z^{(j)})), \qquad i, j = 1,..., m \tag{3.2.8}$$

then each ratio $A_j(z)/G(z), j=1,..., m$ will be single valued, meromorphic for $z \in \mathscr{R}_m$, with poles possible only at the zeros of $G(z)$. This follows since $G(z)$ changes sign whenever $A_j(z)$ does, owing to the interchange of two of its rows.

Now the solution of (3.2.4) shows that

$$\chi_1(z)/A_1(z) = \chi_2(z)/A_2(z) = \cdots = \chi_m(z)/A_m(z), \qquad z \in \mathscr{R}_m \quad (3.2.9)$$

Consequently there is a function $\chi(z)$, meromorphic for $z \in \mathscr{R}_m$, such that

$$\chi_j(z) = \chi(z)\, A_j(z)/G(z), \qquad j=1,..., m,\ z \in \mathscr{R}_m. \quad (3.2.10)$$

The boundary condition (3.2.5) implies

$$(D(z)/G(z-))\,\chi(z-) = R(z+), \qquad z \in s. \quad (3.2.11)$$

This is the standard form of the homogeneous Hilbert problem on $\mathscr{R}$ for functions $R(z)$, $\chi(z)$ meromorphic on $\mathscr{R}_0$, $\mathscr{R}_m$, respectively. In Section 4 we have shown how to use the method of Koppelman [24] to solve such problems.

For polynomials of degree slightly different from n and functions $f_j(z)$ meromorphic, $z \in \mathscr{R}_0$, we expect that the asymptotic form will also correspond to a solution of (3.2.4), (3.2.5).

Now we come to type II polynomials, assuming the same conditions on the functions $f_j(z)$. Again we restrict attention to the diagonal case and take each ρ_j, $j=1,..., m$, of (1.3.5) equal to n, so that $\sigma = mn$ and degree $Q_j(\rho, x) = (m-1)n$, $j=1,..., m$. It is often convenient to introduce polynomials $q_j(z), j=1,..., m$, of degree $(m-1)n$

$$Q_j(\rho, x) = x^{(m-1)n}\, q_j(z), \qquad j=1,..., m. \quad (3.2.12)$$

In this case the conjecture is stated in terms of a function $T(z)$, meromorphic $z \in \mathscr{R}_0$, and a function $\psi(z)$, meromorphic $z \in \mathscr{R}_m$ that satisfy the boundary condition

$$(G(z-)/D(z))\,\psi(z-) = T(z+), \qquad z \in s'. \quad (3.2.13)$$

Now we predict

$$q_j(z) \underset{n\to\infty}{\sim} f_j(z^{(1)})\, T(z^{(1)}), \qquad j=1,..., m,\ z \notin S' \quad (3.2.14)$$

except near zeros of the right-hand side, and $q_j(z)$ will have a zero near those points. We also predict

$$q_j(z) \underset{n\to\infty}{\sim} f_j(z^{(1)})\, T(z^{(1)}) + f_j(z^{(2)})\, T(z^{(2)}), \qquad j=1,\dots,m,\ z\in S' \quad (3.2.15)$$

except possibly near points $z \in S$ or zeros of the right-hand side.

As before, to complete the conjecture we must give enough information about zeros and poles of $\psi(z)$, $T(z)$ to make the solution of (3.2.13) unique, and again we do not give a general statement of this information. However, we do prescribe the following.

$$\begin{aligned} T(z),\ z\in\mathscr{R}_0:\quad & \text{zeros: } (\infty^{(2)})^{n+1},\dots,(\infty^{(m-1)})^{n+1} \\ & \text{poles: } (\infty^{(1)})^{(m-1)n}, \\ & \qquad \{\text{at each branch point of } \mathscr{R}\in\mathscr{R}_0 \text{ a pole} \\ & \qquad \text{of order one less than the winding number}\} \\ \psi(z),\ z\in\mathscr{R}_m:\quad & \text{zeros: } (\infty^{(m)})^{n+1},\ \{\text{poles of } G(z)\} \qquad (3.2.16) \\ & \text{poles: } \{\text{zeros of } G(z)\}. \end{aligned}$$

We see that the solution of (3.2.13) will be the inverse of the solution of (3.2.10) multiplied by a meromorphic function with a number of zeros, poles independent of n. The dominant factor of each $q_j(z)$ for large n is $\exp(-n\phi(z^{(1)}))$ away from S'. Apart from a limited number, the zeros of $q_j(z)$ will approach S' and there will have line density proportional to $|\phi'(z^{(1)}) - \phi'(z^{(2)})|$

3.3. *Conjecture—Case* 2

The evidence of a few examples shows that there may be cases where (3.2.4), (3.2.5) hold, even though the functions $\{f_j(z)\}$ have branch points in $\mathscr{R}_0$. We do not at present have enough information to be very definite about the form of the conjecture in this case, but, in the interest of stimulating further investigation, we make some suggestions.

Suppose that $f_j(z)$, $j=1,\dots,m$, have a branch point at $z=v\in\mathscr{R}_0$. We assume that $v\notin$ sheet 1, although it could be that $v\in s'$, the boundary between sheets 1, 2. We construct a cut $\sigma\in\mathscr{R}_0$, an arc running from v to a point on s, and assume that $\{f_j(z)\}$ are analytic, single valued for $z\in\mathscr{R}_0-\sigma$. The cut σ must be chosen so that as we move along σ from v, the number of the sheet we are on never decreases. No doubt also σ must be a progressive path, one on which $\operatorname{Re}\phi(z)$ does not decrease. Suppose that $v\in$ sheet l, $1<l<m$, or that v is on the boundary of sheet l and not on the boundary of any sheet of higher number. In addition to condition (ii) of Section 3.2, we assume that the limits of $f_j(z)$, $j=1,\dots,m$, exist as either side of σ is approached.

Now suppose that $z^{(l)} \in \sigma$ with $z^{(l)}$ near v. We require the functions $\{f_j(z)\}$ to satisfy

$$f_j(z^{(l)}-) = \lambda(z^{(l)}) f_j(z^{(l)}+) + \sum_{i=1}^{l-1} \lambda_i(z^{(l)}) f_j(z^{(i)}), \qquad j = 1,..., m. \tag{3.3.1}$$

For type I polynomials obeying (3.2.3) we again conjecture that their asymptotic form will be given by a solution of (3.2.4), (3.2.5). The functon $R(z)$ is meromorphic $z \in \mathscr{R}_0 - \sigma$. In (3.2.4), (3.2.5) we must take $\{f_j(z)\}$ to be evaluated in $R_0 - \sigma$ as above, and (3.2.4) also holds as $z^{(k)} \to \sigma$ from either side. A further condition, which makes the boundary value problem have a solution unique up to a meromorphic factor, must also be imposed:

$$R(z-)/R(z+) = \lambda(z), \qquad z \in \sigma. \tag{3.3.2}$$

We expect that, if condition (iii) of Section 3.2 applies, the asymptotic form of $\{p_j(z)\}$ will be given by (3.2.6), (3.2.7), where $\{\chi_j(z)\}$ are the solutions of the boundary problem just described.

We now remark on the solution of this problem. First of all the function $\lambda(z^{(l)})$ may be written as the ratio of two determinants by solving (3.3.1). If we let the point $z = z^{(l)}$move along σ to s, the points $z^{(i)}$, $i = 1,..., l-1$, could well change sheets. However, none of these points will leave $\mathscr{R}_0$ and so $\lambda(z)$ is well defined, $z \in \sigma$. (We assume that σ is chosen so that on no sheet is there a point on σ which is the image of v.)

In place of (3.2.8) we define $G(z)$, piecewise analytic, $z \in \mathscr{R}_m$, by

$$G(z^{(m)}) = \begin{vmatrix} f_1(z^{(1)}) & \cdots & f_m(z^{(1)}) \\ f_1(z^{(2)}) & \cdots & f_m(z^{(2)}) \\ \vdots & & \vdots \\ f_1(z^{(m-1)}) & \cdots & f_m(z^{(m-1)}) \\ g_1(z^{(m)}) & \cdots & g_m(z^{(m)}) \end{vmatrix} \tag{3.3.3}$$

where $g_j(z)$, $j = 1,..., m$ are independent analytic functions, $z \in \mathscr{R}_m$. Indeed, we could replace the $G(z)$ of Section 3.2 by such a formula. Now again (3.2.9) holds and there will be a function $\chi(z)$, meromorphic $z \in \mathscr{R}_m$, such that (3.2.10) is true, with $G(z)$ given by (3.3.3). This follows because (3.3.1) or its continuation may be used to show that $A_j(z)/G(z)$ is single valued, $z \in \mathscr{R}_m$. As before we deduce (3.2.11), still using the new definition of $G(z)$. We point out that $D(z)$, $z \in s$, has a discontinuity at the point where σ joins s, but the continuation of (3.3.1) shows that this discontinuity is consistent with (3.3.2).

Equations (3.2.11), (3.3.2) constitute a boundary value problem which could be solved by the methods of Muskhelishvili [29] if the genus of $\mathscr{R}$ is zero. It is likely that Koppelman's analysis [24] could be extended to treat the case of genus > 0.

For type II polynomials the conjecture of Section 3.2 is modified in an analogous manner. The function $T(z)$ is now meromorphic $z \in \mathscr{R}_0 - \sigma$ and it obeys (3.2.13) with $G(z)$ given by (3.3.3). We also require

$$T(z+)/T(z-) = \lambda(z), \qquad z \in \sigma. \tag{3.3.4}$$

With this $T(z)$ we expect (3.2.14), (3.2.15) to hold.

The example of Section 4.7 corresponds to this case of the conjecture. It is quite possible that there are extensions. A trivial one is to have several branch points v. Another possibility, for which we have no examples, is the situation in which non-trivial cycles exist on $\mathscr{R}_0$ with $\{f_j(z)\}$ consequently not single valued, but a relation, analogous to (3.3.1), holding between the different evaluations of $\{f_j(z)\}$ so that (3.2.4) has a unique solution.

It is also possible, as we see from the second example of Section 5.3, that one of the functions $f_j(z)$ has a branch point on sheet 1. This case does not fit in the previous discussion of this section, and again this indicates that the conjecture might apply to $\{f_j(z)\}$ multi-valued on $\mathscr{R}_0$ obeying a condition other than (3.3.1).

3.4. *Choice of Riemann Surface*

Usually, we begin with a set of functions $\{f_j(z)\}$ and wish to obtain the asymptotic behavior of the corresponding H–P polynomials. In order to do this, it is necessary to determine the appropriate Riemann surface $\mathscr{R}$ to be used in the conjecture. If the conjecture is correct, then there can be at most one surface $\mathscr{R}$ for which the conditions of the conjecture, either case 1 or case 2, hold. Of course, there may be functions, with natural boundaries for instance, for which the conjecture does not apply with any choice of $\mathscr{R}$.

In this section, we study the problem for the case $m = 2$, and show that indeed the surface $\mathscr{R}$ is unique, assuming that $\{f_j(z)\}$ are such that $\mathscr{R}$ exists. A characterization of $\mathscr{R}$ and the corresponding set S is given and information on how they are to be found is presented. We expect that the situation for $m > 2$ will be similar, but so far no progress has been made on its analysis.

With no significant loss of generality, we assume that $f_1(z) = 1$, in which case the determinant D of (3.2.1) becomes

$$D(z^{(2)}) = f_2(z^{(2)}+) - f_2(z^{(1)}+), \qquad z^{(2)} \in s. \tag{3.4.1}$$

The surface $\mathscr{R}$ must consist of two copies of the complex plane cut along S, joined together at this curve. We may write (3.4.1) as

$$D(z^{(2)}) = f_{2+}(z^{(1)}) - f_{2-}(z^{(1)}), \qquad z \in S, \tag{3.4.2}$$

where $+$, $-$ mean the limits from opposite sides of S.

The conditions of Section 3.2 in this case imply that $f_2(z)$ is analytic (on sheet 1) and single valued in the complex plane cut along S, so that $f_2(z)$ must be unchanged as z follows any loop containing one or more components of S. Such loops, not containing the whole of S, are cycles contained in $\mathscr{R}_0$. Case 1a corresponds to the situation when S has but one component. It is also required, as we see from (3.4.2), that the discontinuity of $f_2(z)$ across S is not zero except perhaps at a finite number of points.

A surface $\mathscr{R}$ with two sheets and genus $(l-1)$ may be described by an equation of the form

$$y^2 = X(z) \tag{3.4.3}$$

with

$$X(z) = \prod_{j=1}^{2l} (z - b_j) \tag{3.4.4}$$

where b_j, $j = 1,\dots, 2l$, are distinct, finite points in the complex plane. In [31] we showed that the function $\phi(z)$ of Section 3.1 has the form

$$\phi(z) = \int_{b_1}^{z} dz'\, Y(z')\, X^{-1/2}(z') \tag{3.4.5}$$

where $Y(z)$ is a monic polynomial of degree $l-1$ and $X^{-1/2}(z)$ is the meromorphic function that approaches z^{-l} as $z \to \infty^{(1)}$. The coefficients in $Y(z)$ are determined from the condition that all the periods of $\phi(z)$ are pure imaginary, giving independent equations

$$\operatorname{Re}\left\{\int_{b_1}^{b_{j+1}} dz'\, Y(z')\, X^{-1/2}(z')\right\} = 0, \qquad j = 1,\dots, 2(l-1). \tag{3.4.6}$$

It does not matter which contours are taken in these integrals. If we write $Y(z)$ as

$$Y(z) = z^{l-1} + \sum_{k=1}^{l-1} z^{k-1} y_k, \tag{3.4.7}$$

then (3.4.6) becomes

$$\sum_{k=1}^{l-1} \operatorname{Re}(\Omega_{kj} y_k) = -2 \operatorname{Re}\left\{\int_{b_1}^{b_{j+1}} dz' (z')^{l-1} X^{-1/2}(z')\right\}, \qquad j = 1,\dots, 2(l-1) \tag{3.4.8}$$

where the period matrix Ω_{kj}, made up of period of the integrals of the first kind (in this case $dw_k = z^{k-1}X^{-1/2}(z)\,dz$), is

$$\Omega_{kj} = 2\int_{b_1}^{b_{j+1}} dz'(z')^{k-1}\, X^{-1/2}(z'), \qquad j = 1,\dots, 2(l-1);\ k = 1,\dots, l-1. \tag{3.4.9}$$

The properties of the period matrix [42] ensure that the matrix of (3.4.8) is non-singular and that $Y(z)$ is determined uniquely given $\{b_j\}$.

In this case we have $\operatorname{Re}\phi(z^{(1)}) = -\operatorname{Re}\phi(z^{(2)})$ so that $S = \{z \in \mathbb{C}: \operatorname{Re}\phi(z) = 0\}$. Some information about the set S was given in [31]. It consists of a number of analytic arcs ending at the points $\{b_j\}$. If a zero of $Y(z)$ of multiplicity q coincides with b_k, then $2q+1$ arcs and at b_k, but if the zero belongs to S and is not coincident with any b_j, $j = 1,\dots, 2l$, then $q+1$ arcs intersect at the zero. The set S consists of one or more components. In the complement of S, which is connected, $\operatorname{Re}\phi(z)$ is single valued.

Let us suppose that $b_1,\dots, b_\eta$ are the only points in $\{b_j\}$ that do not coincide with a zero of $Y(z)$, and call these points the ends of S. In the vicinity of one of the ends, the function $f_2(z)$ is analytic apart from a cut running from the end along the arc of S. The discontinuity of $f_2(z)$ across this cut is non-zero except perhaps at a finite number of points. Stretching the meaning of the term, we can say then that this end is a branch point of $f_2(z)$.

Thus, given $f_2(z)$, to determine the appropriate S, and hence immediately $\phi(z)$ and $\mathscr{R}$, we choose a subset of the branch points of $f_2(z)$ to be the ends of S. We group the ends into subsets, each to be contained in the same components of S. As we have shown [36], there is a unique set S corresponding to these requirements, which may described as the set of minimum capacity containing in its components the ends grouped as specified. We now test to see whether the set S constructed in this way meets the requirements of the conjecture that the discontinuity of $f_2(z)$ is not zero (except for a few points) across each arc of S and that it is single valued in the complement of S.

We now show that, no matter how the ends are chosen from the branch points of $f_2(z)$ and how they are then grouped into subsets, there can be at most one set S satisfying the conditions of the conjecture for given $f_2(z)$. We use the method of Grötzsch [18] repeated in [36] for the multi-component case. Suppose that S_1 is a set S for the function $f_2(z)$. Then it corresponds to a $\phi(z)$ of the form described above. The loci $\operatorname{Im}\phi(z) = \text{const.}$ are non-intersecting analytic arcs each running from a point on S_1 to ∞. Arcs leaving S_1 from opposite sides may be paired to form a single arc, which we call a curve, running from ∞ to ∞ and intersecting S_1 once. For S_2, another possible set S for $f_2(z)$, we construct a similar set of curves.

We discuss two possibilities.

(i) For at least one set, say S_1, there is a curve of S_1 that does not intersect S_2, or

(ii) All the curves of S_1 intersect S_2 and vice versa.

The first possibility can be ruled out since there must be a continuum of curves of S_1 near to the given curve that do not intersect S_2. Because S_2 is a possible set S the change in $f_2(z)$ as we follow one of these curves from ∞ to ∞ must be zero, but, since the curves cross S_1, this cannot be so, a contradiction. If the second possibility were to hold, the argument of Grötzsch [18] would show that the capacity of S_2 was greater than the capacity of S_1, and vice versa, also a contradiction. In this way the uniqueness of S is demonstrated.

4. Rigorous Results

In a number of special cases, rigorous results on the asymptotic behavior of Hermite–Padé polynomials have been obtained. The conjecture has been designed to fit all these results. This section contains a summary of many of the rigorous results, including new work, for which more details are given. For ease of exposition we usually restrict the discussion to the diagonal case, but many of the results could be extended to the near-diagonal case with little difficulty.

4.1. *Meromorphic Functions—Type I Polynomials*

A case of basic interest is that in which we are given a Riemann surface $\mathscr{R}$ with m sheets as in Section 3.1, and a set of functions $f_i(z)$, $i = 1,..., m$, which are meromorphic on $\mathscr{R}$, with poles restricted to $\mathscr{R}_m$. This means to say that each function $f_i(z)$ is a rational function of y, z, which are related by (3.1.1).

A sheet structure is introduced on $\mathscr{R}$ as in Section 3.1. We assume that each $f_i(z)$ is analytic in the neighborhood of $\infty^{(1)}$, the point of expansion (assumed not to be a branch point of $\mathscr{R}$), and that $D(z)$ given by (3.2.1) (in this case defined for all $z \in \mathscr{R}_m$) is not identically zero. For simplicity, we also assume that no $\infty^{(j)}$, $j = 2,.., m$, is a branch point of $\mathscr{R}$. It follows that the set of functions $f_i(z)$ satisfies the conditions (i)–(iii) of Section 3.2.

The case of type I polynomials was first treated by Nuttall [34] and this section extends the previous results. The H–P polynomials defined in Section 3.2 by (3.2.3) may not be unique, but for any possible set of degree n we set

$$\sum_{j=1}^{m} f_j(z)\, p_j(z) = \bar{R}(z) \tag{4.1.1}$$

where

$$\bar{R}(z) = O(z^{-(m-1)(n+1)}) \qquad \text{near } z = \infty^{(1)}. \tag{4.1.2}$$

We suppose that the poles of all $f_i(z)$ are contained in the set of finite points $a_j, j = 1,..., \lambda$, $a_j \in \mathscr{R}_m$, so that $\bar{R}(z)$ is meromorphic on $\mathscr{R}$ with poles possible only at $\{a_j\}$ and $\infty^{(k)}$, $k = 2,..., m$, the latter poles being of order n.

Now in (34) it was shown that meromorphic functions $H_i(z)$ exist such that

$$\sum_{i=1}^{m} f_i(z^{(k)}) H_i(z^{(j)}) = \delta_{kj}, \qquad j, k = 1,..., m \tag{4.1.3}$$

so that we can write (4.1.1) for $z = z^{(k)}$, $k = 1,..., m$, solve, and obtain

$$p_j(z) = \sum_{k=1}^{m} \bar{\chi}_j(z^{(k)}), \qquad j = 1,..., m. \tag{4.1.4}$$

We have set

$$\bar{\chi}_j(z) = H_j(z)\bar{R}(z) \tag{4.1.5}$$

so that $\bar{\chi}_j(z)$ is meromorphic, $z \in \mathscr{R}$.

For a meromorphic function such as $\bar{R}(z)$ the number of zeros equals the number of poles [42], so that in addition to a zero of order $(m-1)(n+1)$ at $\infty^{(1)}$, there must be other zeros $\bar{c}_j$, $j = 1,..., \lambda - m + 1$. We set $\bar{c}_j = \infty^{(1)}$, $j = \lambda - m + 2,..., \lambda$. It was shown in [34] that $\bar{R}(z)$ could be written in the form (up to a constant factor)

$$\bar{R}(z) = \bar{\zeta}(z) \exp(n\phi(z)) \tag{4.1.6}$$

where

$$\bar{\zeta}(z) = \exp\left(\sum_{j=1}^{\lambda} E(\bar{c}_j, a_j; z)\right) \tag{4.1.7}$$

and $\phi(z)$ is the function of Section 3.1, which may be written explicitly as

$$\phi(z) = \sum_{j=2}^{m} E(\infty^{(1)}, \infty^{(j)}; z). \tag{4.1.8}$$

We have used the unique differential $dE(z_1, z_2)$ of the third kind, whose singularities are simple poles at z_1, z_2 with residues $1, -1$, respectively, such that the periods of the integral $E(z_1, z_2; z)$ are pure imaginary [42]. The function $\exp(E(z_1, z_2; z))$ has a zero at $z = z_1$, and a pole at $z = z_2$.

We now present a theorem about the asymptotic behavior of $p_j(z)$, a result which agrees with the conjecture of Section 3. For this purpose, we restrict attention to the general case, described in the following.

In general, the branch points of $\mathscr{R}$, that is solutions of

$$\begin{aligned} r(y, z) &= 0 \\ \frac{\partial r}{\partial y}(y, z) &= 0 \end{aligned} \tag{4.1.9}$$

will be square root branch points. We assume the surface is such that all branch points are of square root type, so that no value of $z \in \mathbb{C}$ exists for which (4.1.9) has a root y of third of higher order. We also assume that all the poles of $\{f_j(z)\}$ are simple and that $D(z) = \det(f_j(z^{(k)}))$, $z \in \mathscr{R}_m$, has a simple pole at each point a_j, $j = 1,..., \lambda$.

Now it is seen that $D^2(z)$ is analytic, $z \in \mathbb{C}$, except for second-order poles at the image of each a_j, $j = 1,..., \lambda$. This function has zeros, assumed simple, at the image of every branch point of $\mathscr{R}$. The number of branch points is $v = 2(g + m - 1)$ (Siegel [41]), where g is the genus of $\mathscr{R}$ so that $D(z)$ has an additional $[\lambda - (g + m) + 1) + 1]$ zeros. We shall assume that the images of these zeros are all finite and do not lie on S. We let $c_j \in \mathscr{R}_m$, $j = g + 1,...,$ $\lambda - m + 1$, be the additional zeros of $D(z)$, with images assumed to be distinct from those of the branch points. The function $H_j(z^{(m)})$ is the ratio of a cofactor to $D(z)$, and we assume that for i, $i = g + 1,..., \lambda - m + 1$, at least one of $H_j(z^{(m)})$, $j = 1,..., m$, has a pole at c_i. Finally, we assume that no branch point has an image lying on S, which implies $m > 2$.

With these assumptions, the theorem may be stated as

THEOREM 4.1. *Let the points $c_j \in \mathscr{R}$, $j = 1,..., g$, depending on n, be chosen so that a meromophic function $R(z)$ exists with the following zeros, poles.*

$$R(z){:}\quad \begin{aligned} &\text{zeros: } (\infty^{(1)})^{(m-1)(n+1)},\ c_{g+1},..., c_{\lambda-m+1},\ c_1,..., c_g \\ &\text{poles: } (\infty^{(2)})^n,..., (\infty^{(m)})^n,\ a_1,..., a_\lambda. \end{aligned} \tag{4.1.10}$$

Suppose that the divisor $c_1 \cdots c_g$ is not within a given small distance (using a topology provided by the local variable on $\mathscr{R}$) of any special divisor and that no c_i, $i = 1,..., g$, is near to $\infty^{(1)}$. Then, for sufficiently large n, the point $\rho_j = n + 1$, $j = 1,..., m$, is normal (see Section 1.3) so that the polynomials $p_j(z)$ are unique, and, subject to the remark below,

$$p_j(z) \underset{n \to \infty}{\sim} \chi_j(z^{(m)}), \qquad j = 1,..., m,\ z \notin S \tag{4.1.11}$$

where the function $\chi_j(z)$, *analytic for* $z \in \mathscr{R}_m$ *except for a pole of order* n *at* $z = \infty^{(m)}$, *is given by*

$$\chi_j(z) = H_j(z)\, R(z) = H_j(z)\, \zeta(z) \exp(n\phi(z)) \tag{4.1.12}$$

with

$$\zeta(z) = \exp\left[\sum_{j=1}^{\lambda-m+1} E(c_j, a_j; z) + \sum_{j=\lambda-m+2}^{\lambda} E(\bar{c}_j, a_j; z)\right] \tag{4.1.13}$$

Near to any c_k, $k = 1, \ldots, g$ *for which* $c_k \in \mathscr{R}_m$, (4.1.11) *does not hold. For such points* $\chi_j(c_k) = 0$, $j = 1, \ldots, m$, *and* $p_j(z)$ *will have a nearby zero. We also have*

$$p_j(z) \underset{n\to\infty}{\sim} \chi_j(z^{(m)}) + \chi_j(z^{(m-1)}), \qquad j = 1, \ldots, m,\ z \in S \tag{4.1.14}$$

except near zeros of the right-hand side.

Proof. A proof may be constructed based on the following points.

1. For $z \notin S$, $\operatorname{Re} \phi(z^{(m)}) > \operatorname{Re} \phi(z^{(m-1)})$ so that

$$\exp(n\phi(z^{(m)})) \gg \exp(n\phi(z^{(m-1)})) \text{ as } n \to \infty, \qquad z \notin S. \tag{4.1.15}$$

2. At least one of $\bar{\chi}_j(z^{(m)})$, $j = 1, \ldots, m$, will have a pole at $z^{(m)} = c_k$, $k = g+1, \ldots, \lambda - m + 1$, but because the sum (4.1.4) must be analytic, (4.1.15) implies that a zero of $R(z)$ must lie close to each such c_k. Thus, say,

$$\bar{c}_k \approx c_k,\ k = g+1, \ldots, \lambda - m + 1. \tag{4.1.16}$$

3. From the definition (4.1.10) we see that the meromorphic function $R(z)/\bar{R}(z)$ has zeros, poles as follows

$$R(z)/\bar{R}(z): \quad \begin{aligned} &\text{zeros: } c_1, \ldots, c_{\lambda-m+1} \\ &\text{poles: } \bar{c}_1, \ldots, \bar{c}_{\lambda-m+1}. \end{aligned} \tag{4.1.17}$$

Thus from (3.1.6)

$$\sum_{j=1}^{g} \int_{c_j}^{\bar{c}_j} dw_k = -\sum_{j=g+1}^{\lambda-m+1} \int_{c_j}^{\bar{c}_j} dw_k + \sum_{i=1}^{2g} n_i \Omega_{ki}, \qquad k = 1, \ldots, g. \tag{4.1.18}$$

From (4.1.16) the right-hand side of (4.1.18) may be made to approach zero for each k, from which it follows [42], with the help of the assumption about the divisor $c_1 \cdots c_g$, that, for large n

$$\bar{c}_j \approx c_j, j = 1,..., g. \tag{4.1.19}$$

The theorem follows.

We expect that an analogous theorem will hold when the restrictions on $\mathscr{R}$ and the meromorphic functions $\{f_j(z)\}$ are removed.

Now let us consider how the asymptotic behavior given in this theorem relates to the conjecture of Section 3.2. Clearly the meromorphic functions $\chi_j(z)$, $z \in \mathscr{R}_m$, and $R(z)$, $z \in \mathscr{R}_0$, satisfy Eqs. (3.2.4), (3.2.5). We shall now illustrate the general method of solving the equations of the conjecture. In this case, to make the conjecture complete, we specify that the functions $R(z)$, $\chi_j(z)$ must have zeros, poles as follows.

$$\begin{aligned} \chi_j(z),\ z \in \mathscr{R}_m&: \quad \text{poles: } (\infty^{(m)})^n \\ R(z),\ z \in \mathscr{R}_0&: \quad \text{zeros: } (\infty^{(1)})^{(m-1)(n+1)} \\ &\quad\ \ \text{poles: } (\infty^{(2)})^n,..., (\infty^{(m-1)})^n. \end{aligned} \tag{4.1.20}$$

The functions have no other poles and have limits on s which are smooth, but they may have other zeros.

Now we follow the procedure of Section 3.2. to solve (3.2.4), (3.2.5), and introduce the determinant $G(z)$ with properties similar to those of $D(z)$. Thus $G^2(z)$ will have zeros, which we assume to be simple, at the image of each branch point of $\mathscr{R}$. In addition $G(z)$ will have zeros, poles at $\gamma_1,..., \gamma_v$ and $\alpha_1,..., \alpha_\eta$, respectively, and we assume that none of these points lies on S.

Our aim is to solve (3.2.11), where $R(z)$ has zeros, poles in $\mathscr{R}_0$ as in (4.1.20), and, from (3.2.10), $\chi(z)$ is meromorphic $z \in \mathscr{R}_m$ with zeros, poles prescribed as

$$\begin{aligned} \chi(z),\ z \in \mathscr{R}_m&: \quad \text{zeros: } \gamma_1^{(m)},..., \gamma_v^{(m)} \\ &\quad\ \ \text{poles: } (\infty^{(m)})^n,\ \alpha_1^{(m)},..., \alpha_\eta^{(m)} \end{aligned} \tag{4.1.21}$$

The method of Koppelman [24] requires the index κ defined by

$$\kappa = \frac{1}{2\pi} \left[\arg(D(z)/G(z))\right]_s. \tag{4.1.22}$$

We note that $D(z)/G(z)$ is meromorphic for $z \in \mathscr{R}_m$ so that

$$\begin{aligned} \kappa = {} & (\text{No. of poles of } D(z) \in \mathscr{R}_m) - (\text{No. of zeros of } D(z) \in \mathscr{R}_m) \\ & + (\text{No. of zeros of } G(z) \in \mathscr{R}_m) - (\text{No. of poles of } G(z) \in \mathscr{R}_m) \end{aligned}$$

i.e.,

$$\kappa = g + m - 1 + \nu - \eta. \tag{4.1.23}$$

Now Koppelman describes how to construct a meromorphic function $J(z)$, $z \in \mathscr{R}_0$, such that

$$[\arg J(z+)]_{s_i} = -[\arg(D(z)/G(z))]_{s_i}, \text{ each } i. \tag{4.1.24}$$

The function $J(z)$ has no poles, zeros, for $z \in \mathscr{R}_0$ except for a pole of order κ at a chosen point $z_0 \in \mathscr{R}_0$ (a zero if $\kappa < 0$). If we introduce $\Omega(z)$, meromorphic for $z \in \mathscr{R} - s$, by

$$\begin{aligned} \Omega(z) &= \chi(z), & z &\in \mathscr{R}_m \\ \Omega(z) &= R(z)\, J(z), & z &\in \mathscr{R}_0 \end{aligned} \tag{4.1.25}$$

then

$$\Omega(z+) = \rho(z)\, \Omega(z-), \qquad z \in s \tag{4.1.26}$$

where

$$\rho(z) = (D(z)/G(z))\, J(z+), \qquad z \in s \tag{4.1.27}$$

so that

$$[\arg \rho(z)]_{s_i} = 0, \qquad \text{each } i. \tag{4.1.28}$$

The equation

$$\log \Omega(z+) - \log \Omega(z-) = \log \rho(z) \tag{4.1.29}$$

deduced from (4.1.26), is solved with the help of a kernel $K(t, z)$ defined by Koppelman [24]. Indeed, the function

$$\theta_1(z) = \frac{1}{2\pi i} \int_s dt\, \frac{\partial K(t, z)}{\partial t} \log \rho(t) \tag{4.1.30}$$

is meromorphic, $z \in \mathscr{R} - s$, with poles at chosen points z_j, $j = 1,..., g$, and has discontinuity $\log \rho(z)$ across s. We add $\theta_2(z)$, an integral of the second kind (Siegel [42]), to $\theta_1(z)$ to cancel the poles, but the periods of $\theta_2(z)$ will not in general vanish.

To proceed, we use a lemma, almost identical to one given by Koppelman, and easily proved with the help of results in his paper.

LEMMA 4.2. *For $k \geqslant g$, given any α_j, $j = 1,..., k$, and β_j, $j = g + 1,..., k$,* *on $\mathscr{R}$, it is possible to find $\beta_j \in \mathscr{R}$, $j = 1,..., g$, and an integral of the first kind $w(z)$ such that the periods of*

$$\theta_3(z) = \sum_{j=1}^{k} E(\beta_j, \alpha_j; z) + w(z) \tag{4.1.31}$$

are any prescribed values.

Now in the case at hand (we are still using the assumptions above Th. 4.1) zeros and poles of $\Omega(z)$ must occur as follows

$$\begin{aligned} \Omega(z),\ z \in \mathscr{R} - s:\quad &\text{zeros: } (\infty^{(1)})^{(m-1)(n+1)},\ \gamma_1^{(m)},..., \gamma_v^{(m)} \\ &\text{poles: } (\infty^{(2)})^n,..., (\infty^{(m)})^n,\ \alpha_1^{(m)},..., \alpha_\eta^{(m)},\ (z_0)^\kappa. \end{aligned} \tag{4.1.32}$$

It follows that the number of poles listed in (4.1.32) exceeds the number of zeros by g, and we use the listed poles and zeros plus g additional points $\beta_j \in \mathscr{R}$, $j = 1,..., g$, to construct $\theta_3(z)$ with periods that cancel those of $\theta_2(z)$. The result is the solution

$$\Omega(z) = \exp(\theta_1(z) + \theta_2(z) + \theta_3(z)) \tag{4.1.33}$$

which the construction shows to be unique unless the points $\beta_1,..., \beta_g$ correspond to a special divisor.

Now of course in this case the function $R(z)$, $z \in \mathscr{R}_0$, obtained by this procedure may be extended to a function meromorphic for $z \in \mathscr{R}$, namely the function specified by (4.1.10). It is seen that $\beta_1 \cdots \beta_g = c_1 \cdots c_g$. By solving the equations of the conjecture, with appropriate specifications of zeros, poles, we are able to predict that a point $\rho_j = n + 1$, $j = 1,..., m$ is normal for large enough n by checking that the divisor $\beta_1 \cdots \beta_g$ is not close to special and has no point near $\infty^{(1)}$. If this holds, we can predict the asymptotic behavior of the polynomials $\{p_j(z)\}$.

It is our expectation that the same situation applies for a wider class of functions $\{f_j(z)\}$. If these functions are analytic for $z \in \mathscr{R}_0$ and have sufficiently smooth limiting values as $z \to s$, with $D(z) \neq 0$, $z \in s$, then the method of solving the conjecture may be used word for word. We predict that the results about normality and asymptotic behavior will be the same.

4.2. *Meromorphic Functions—Type II Polynomials*

With the notation (3.2.12) of Section 3.2 the equations defining type II polynomials may, in the diagonal case, be written

$$f_i(z^{(1)})\, q_j(z) - f_j(z^{(1)})\, q_i(z) = O(z^{-(n+1)}), \qquad i, j = 1,..., m \tag{4.2.1}$$

where the polynomials $\{q_j(z)\}$ are of degree $(m-1)n$.

We continue to assume that $\{f_j(z)\}$ are meromorphic and satisfy the conditions of Theorem 4.1. For a given $z \in \mathbb{C}$ we define $\bar{T}_k(z)$ as the solution of

$$\sum_{k=1}^{m} f_j(z^{(k)})\,\bar{T}_k(z) = q_j(z), \qquad j = 1,\dots, m. \tag{4.2.2}$$

With the help of (4.1.3) we find

$$\bar{T}_k(z) = \sum_{j=1}^{m} H_j(z^{(k)})\,q_j(z) \tag{4.2.3}$$

so that we have

$$\bar{T}_k(z) = \bar{T}(z^{(k)}) \tag{4.2.4}$$

where the meromorphic function $\bar{T}(z)$, $z \in \mathscr{R}$, is given by

$$\bar{T}(z) = \sum_{j=1}^{m} H_j(z)\,q_j(z). \tag{4.2.5}$$

Now, substituting in (4.2.1) we find that, as $z \to \infty$,

$$\sum_{k=2}^{m} [f_i(z^{(1)})\,f_j(z^{(k)}) - f_j(z^{(1)})\,f_i(z^{(k)})]\,\bar{T}(z^{(k)}) = O(z^{-(n+1)}), \qquad i,j = 1,\dots, m. \tag{4.2.6}$$

With the assumption $f_1(\infty^{(1)}) \neq 0$, take (4.2.6) with $i = 1$ to give

$$f_1(z^{(1)}) \left\{ f_j(z^{(1)})\,\bar{T}_0 + \sum_{k=2}^{m} f_j(z^{(k)})\,\bar{T}(z^{(k)}) \right\} = O(z^{-(n+1)}), \qquad j = 2,\dots, m. \tag{4.2.7}$$

We have defined $\bar{T}_0$ by

$$f_1(z^{(1)})\,\bar{T}_0 + \sum_{k=2}^{m} f_1(z^{(k)})\,\bar{T}(z^{(k)}) = 0. \tag{4.2.8}$$

The determinant of the matrix of Eqs. (4.2.7), (4.2.8) for $\bar{T}_0$, $\bar{T}(z^{(k)})$, $k = 2,\dots, m$, is $(f_1(z^{(1)}))^{m-1}\,D(z)$. By assumption this is non-zero at ∞, so that (4.2.6) implies, as $z \to \infty$,

$$\bar{T}(z^{(k)}) = O(z^{-(n+1)}), \qquad k = 2,\dots, m. \tag{4.2.9}$$

From this property and (4.2.5) we see that $\bar{T}(z)$ must have the following zeros and poles on $\mathscr{R}$.

$$\begin{aligned}&\text{zeros: } (\infty^{(2)})^{n+1},\dots,(\infty^{(m)})^{n+1}, a_1,\dots,a_\lambda \\ &\text{poles: } (\infty^{(1)})^{(m-1)n}, \{c_j^{(k)}, k=1,\dots,m; j=g+1,\dots,\lambda-m+1\}, b_1,\dots,b_\nu.\end{aligned} \tag{4.2.10}$$

By $c_j^{(k)}$ we mean the image on sheet k of $c_j \in \mathscr{R}_m$ corresponding to zeros of $D(z)$. The points $b_j \in \mathscr{R}_0, j=1,\dots,\nu$, are the branch points of $\mathscr{R}$. We remark that a simple pole at a branch point is not inconsistent with (4.2.2), which is equivalent to

$$q_j(z) = \sum_{k=1}^{m} f_j(z^{(k)})\,\bar{T}(z^{(k)}), \tag{4.2.11}$$

for the singular contributes from two terms in the sum will cancel.

Remembering that $\nu = 2(g+m-1)$, we see that there must be an additional $[(m-1)(\lambda-(g+m)+1)+g]$ zeros in $\bar{T}(z)$. We follow a procedure similar to that of Section 4.1 and obtain an expression for $\bar{T}(z)$ containing the factor $\exp(-n\phi(z))$, which, for large n, is largest on sheet 1, smallest on sheet m. Because the residues of poles at $c_j^{(k)}$, $k=1,\dots,m$, must cancel in (4.2.11), we deduce, as before, that, for large n, zeros of $\bar{T}(z)$ must occur near $c_j^{(k)}$, $k=1,\dots,m-1; j=g+1,\dots,\lambda-m+1$. As a result, we have the following theorem, analogous to Theorem 4.1.

THEOREM 4.3. *Make the same assumptions as are required for Theorem 4.1. Let the points $t_j \in \mathscr{R}$, $j=1,\dots,g$, depending on n, be chosen so that a meromorphic function $T(z)$ exists with the following zeros, poles.*

$$\begin{aligned} T(z):\quad &\textit{zeros}: (\infty^{(2)})^{n+1},\dots,(\infty^{(m)})^{n+1}, a_1,\dots,a_\lambda, t_1,\dots,t_g \\ &\textit{poles}: (\infty^{(1)})^{(m-1)n}, c_{g+1},\dots,c_{\lambda-m+1}, b_1,\dots,b_\nu \end{aligned} \tag{4.2.12}$$

Suppose that the divisor $t_1 \cdots t_g$ is not near to any special divisor and no t_i, $i=1,\dots,g$ is near to $\infty^{(1)}$. Then for sufficiently large n, the point $\rho_j = n$, $j=1,\dots,m$, is normal so that the polynomials $q_j(z)$ are unique and $x^{(m-1)n}q_1(z) \neq 0$ at $x=0$. The asymptotic behavior is given by

$$q_j(z) \underset{n\to\infty}{\sim} f_j(z^{(1)})\,T(z^{(1)}), \qquad j=1,\dots,m,\ z\in S' \tag{4.2.13}$$

except near to a zero of the right-hand side, near which point $q_j(z)$ will be zero. We also have

$$q_j(z) \underset{n\to\infty}{\sim} f_j(z^{(1)})\,T(z^{(1)}) + f_j(z^{(2)})\,T(z^{(2)}), \qquad j=1,\dots,m,\ z\in S'. \tag{4.2.14}$$

To obtain a function $\psi(z)$ that, together with $T(z)$, $z\in\mathscr{R}_0$, satisfies (3.2.13) of the conjecture, we define

$$\psi(z) = (G(z)/D(z))\,T(z), \qquad z\in\mathscr{R}_m. \tag{4.2.15}$$

In this case, all the branch points have winding number 2 and lie in $\mathscr{R}_0$. It is possible to follow the procedure of Koppelman explained in Section 4.1 to see that the specification (3.2.16) leads uniquely to the above solution for the equation of the conjecture, in the case when the divisor $t_1 \cdots t_g$ is not special. The situation about predicting normality is similar to that described in Section 4.1. Again, we expect the results to apply to a wider class of $\{f_j(z)\}$.

4.3. *Special Cases of Meromorphic Functions*

This section studies the construction and asymptotics of H–P polynomials for meromorphic functions in a number of special cases. We begin with $m = 2$, in which case all branch points are of square root type and lie on s, and the two types of polynomial are essentially the same. The Akhiezer polynomials are treated first followed by the important special cases of Bernstein–Szegö and Jacobi–Dumas polynomials. In Section 4.3.4 we discuss the construction of helpful examples when $m > 2$.

4.3.1. *Akhiezer Polynomials.* Akhiezer [1] discovered a set of polynomials orthogonal when integrated along the real axis with a particular weight function that was non-zero for l disjoint intervals. Later, unaware of this work, Nuttall and Singh [31] used the same idea but considered complex branch points and saw how the results pointed to the general structure described in this article.

We take $m = 2$ and consider the surface $\mathscr{R}$ given by (3.4.3) with the corresponding $\phi(z)$ and S described in Section 3.4. Akhiezer [1] studied the case when all b_j are real, say $b_1 < b_2 \cdots < b_{2l}$. The set S is then the line segments $[b_1, b_{j+1}^j]$, $j = 1, 3, \ldots, 2l-1$, and a zero of $Y(z)$ lies in each gap between these segments.

Now to proceed we choose $f_1(z) = 1$ and set (with $+$ meaning the left side of S)

$$f_2(z) = \text{const.} + (2\pi i)^{-1} \int_S d\bar{z}\, X_+^{-1/2}(\bar{z}^{(1)})\, \rho(\bar{z})(\bar{z} - z)^{-1}, \qquad z \in \mathscr{R}_0 \qquad (4.3.1)$$

where

$$\rho(z) = \prod_{j=1}^{\lambda} (z - a_j)^{-1}, \qquad a_j \notin S. \qquad (4.3.2)$$

Thus $f_2(z)$ is meromorphic for $z \in \mathscr{R}$ with simple poles at $z = b_j$, $j = 1, \ldots, 2l$, and $z = a_j^{(2)}$, $j = 1, \ldots, \lambda$. We see that in this case

$$D(z^{(2)}) = f_2(z^{(2)}+) - f_2(z^{(1)}+) = X^{-1/2}(z^{(2)})\, \rho(z), \qquad z^{(2)} \in s, \quad (4.3.3)$$

so that $D(z^{(2)})$ has a meromorphic extension, $z \in \mathscr{R}$,

$$D(z) = X^{-1/2}(z)\, \rho(z). \qquad (4.3.4)$$

Using (4.3.3), we find that the meromorphic function $R(z)$ defined by (4.1.1) may be written on sheet 2 as

$$\begin{aligned}\bar{R}(z^{(2)}) &= p_1(z) + f_2(z^{(2)})\, p_2(z) \\ &= p_1(z) + f_2(z^{(1)})\, p_2(z) + X^{-1/2}(z^{(2)})\, \rho(z)\, p_2(z) \\ &= \bar{R}(z^{(1)}) + X^{-1/2}(z^{(2)})\, \rho(z)\, p_2(z). \end{aligned} \tag{4.3.5}$$

Thus $\bar{R}(z)$ has zeros and poles, supposing $n \geqslant \lambda + l$,

$$\begin{aligned}\bar{R}(z)\colon \quad &\text{zeros: } (\infty^{(1)})^{n+1} \\ &\text{poles: } (\infty^{(2)})^{n-\lambda-l},\ a_1^{(2)},\dots, a_\lambda^{(2)},\ b_1,\dots, b_{2l}.\end{aligned} \tag{4.3.6}$$

There are therefore additional zeros $\bar{c}_j \in \mathscr{R}$, $j = 1,\dots, l-1$ of $\bar{R}(z)$. Since the genus of $\mathscr{R}$ is $l-1$, these points are determined uniquely unless they correspond to a speial divisor.

Having $\bar{R}(z)$, we follow the procedure of Section 4.1 to obtain $p_j(z)$ as

$$p_j(z) = \bar{\chi}_j(z^{(1)}) + \bar{\chi}_j(z^{(2)}), \qquad j = 1, 2 \tag{4.3.7}$$

with $\bar{\chi}_j(z)$ given by (4.1.6). This gives

$$\begin{aligned}\bar{\chi}_2(z) &= D^{-1}(z)\, \bar{R}(z) \\ &= X^{1/2}(z)\, \rho^{-1}(z)\, \bar{R}(z)\end{aligned} \tag{4.3.8}$$

so that $\bar{\chi}_2(z)$ has zeros, poles

$$\begin{aligned}\bar{\chi}_2(z)\colon \quad &\text{zeros: } (\infty^{(1)})^{n+1-\lambda-l},\ a_1^{(1)},\dots, a_\lambda^{(1)},\ \bar{c}_1,\dots, \bar{c}_{l-1} \\ &\text{poles: } (\infty^{(2)})^{n}.\end{aligned} \tag{4.3.9}$$

In [31] it was shown directly that such a $\bar{\chi}_2(z)$ led to $p_2(z)$ that satisfied the appropriate orthogonality conditions.

We also find that

$$\begin{aligned}\bar{\chi}_1(z^{(1)}) &= -f_2(z^{(2)})\, \bar{\chi}_2(z^{(1)}) \\ \bar{\chi}_1(z^{(2)}) &= -f_2(z^{(1)})\, \bar{\chi}_2(z^{(2)})\end{aligned} \tag{4.3.10}$$

so that $\bar{\chi}_1(z)$ has

$$\begin{aligned}\bar{\chi}_1(z)\colon \quad &\text{zeros: } (\infty^{(1)})^{n+1-\lambda-l},\ \bar{c}_1,\dots, \bar{c}_{l-1} \\ &\text{poles: } (\infty^{(2)})^{n},\ b_1,\dots, b_{2l}\end{aligned} \tag{4.3.11}$$

plus some additional zeros.

Thus, for $z \notin S$, $\bar{\chi}_j(z^{(2)})$, $j = 1, 2$, have no poles except at $\infty^{(2)}$, and it is easy to apply the methods of Section 4.1 to obtain:

THEOREM 4.4. *With $f_2(z)$ given by* (4.3.1), *the asymptotic behavior of polynomials $p_1(z)$, $p_2(z)$ of degree n satisfying*

$$p_1(z) + f_2(z^{(1)})\, p_2(z) = O(z^{-(n+1)}) \qquad as \quad n \to \infty \tag{4.3.12}$$

is

$$p_j(z) \underset{n\to\infty}{\sim} \bar{\chi}_j(z^{(2)}), \qquad j = 1, 2, z \notin S \tag{4.3.13}$$

except near zeros of $\bar{\chi}_j(z^{(2)})$, near to which $p_j(z)$ will be zero. For $z \notin S$ the exact form (4.3.7) *applies.*

We see that this theorem holds whether or not $\bar{c}_1 \cdots \bar{c}_{l-1}$ is a special divisor. Should the divisor be special, the polynomials are not unique but the behavior of all possible sets is given.

Now we wish to show how the asymptotic form $\bar{\chi}_j(z)$, which, with $\bar{R}(z)$ from (4.3.6), obey Eqs. (3.2.4), (3.2.5), can be derived uniquely (usually) from the conjecture. As before the equations of the conjecture lead to (3.2.11), and in this case, since each $A_j(z)$ is single valued for $z \in \mathscr{R}_2$, we may choose $G(z) = 1$. Since $A_2(z) = 1$, we have $\chi(z) = \chi_2(z)$ so that the specification of the poles of $\chi(z)$, $z \in \mathscr{R}_2$, is

$$\chi(z)\colon z \in \mathscr{R}_2\colon \quad \text{poles: } (\infty^{(2)})^n. \tag{4.3.14}$$

We see from (4.3.6) that $\bar{R}(z) = R(z)$ has poles at the branch points $b_1, \ldots, b_{2l}$, which lie on S. To obtain a problem to which Koppelman's method applies, we introduce $R^*(z) = X^{1/2}(z)\, R(z)$ and require from (4.3.6) zeros, poles, $z \in \mathscr{R}_0$ as follows

$$\begin{aligned} R^*(z)\colon z \in \mathscr{R}_0\colon \quad & \text{zeros: } (\infty^{(1)})^{n+1-l} \\ & \text{poles: none} \end{aligned} \tag{4.3.15}$$

As we shall see, the requirements (4.3.14), (4.3.15) along with the boundary condition

$$\begin{aligned} & (X^{1/2}(z)\, D(z))\, \chi(z-) = R^*(z+), \qquad z \in s \\ \text{i.e.,} \quad & \rho(z)\, \chi(z-) = R^*(z+); \qquad z \in s \end{aligned} \tag{4.3.16}$$

constitute a well-posed problem with a solution that is usually unique.

This problem could be solved by the general method of Koppelman, but, because $m = 2$, it can be reduced to a problem in the complex plane that can be treated by the techniques of Muskhelishvili [29]. An explicit solution was given in [31], and, because of its importance in connection with the asymptotic behavior in the general Padé case, we sketch a derivation here.

Both $\mathscr{R}_0$ and $\mathscr{R}_2$ are copies of the complex plane cut along S and for the rest of Section 4.3.1 we shall use $R^*(z)$, $z \in \mathbb{C} - S$, to denote $R^*(z^{(1)})$, and similarly $\chi(z)$, $z \in \mathbb{C} - S$, to denote $\chi(z^{(2)})$. Thus (4.3.16) becomes

$$\begin{aligned} \rho(z)\,\chi_+(z) &= R^*_-(z) \\ \rho(z)\,\chi_-(z) &= R^*_+(z) \end{aligned} \qquad z \in S. \tag{4.3.17}$$

Cross-multiplying gives

$$\chi_+(z)\,R^*_+(z) = \chi_-(z)\,R^*_-(z), \qquad z \in S \tag{4.3.18}$$

so that $\chi(z)\,R^*(z)$ must be a polynomial of degree $(l-1)$ from (4.3.14), (4.3.15). We write

$$\chi(z)\,R^*(z) = \prod_{j=1}^{l-1} (z - \alpha_j), \qquad \alpha_j \in \mathbb{C} \tag{4.3.19}$$

and for simplicity of argument assume that no $\alpha_j \in S$. Zeros of the right-hand side can occur if and only if one of the factors $\chi(z)$, $R^*(z)$ has a zero. We suppose that $\alpha_j, j = 1,..., v$ are zeros of $\chi(z)$ and $\alpha_j, j = v+1,..., l-1$, are zeros of $R^*(z)$.

If we set

$$\chi(z) = (z - b_1)^{n-v} \left(\prod_{j=1}^{v} (z - \alpha_j)\right) \xi(z) \tag{4.3.20}$$

then $\xi(z)$ is analytic and non-zero, $z \in \mathbb{C} - S$, but has a pole of order $n - v$ at $z = b_1$. From (4.3.17) and (4.3.19) we find

$$\xi_+(z)\,\xi_-(z) = \rho^{-1}(z) \left(\prod_{j=1}^{v} (z - \alpha_j)^{-1}\right)\left(\prod_{j=v+1}^{l-1} (z - \alpha_j)\right)(z - b_1)^{-2(n-v)},$$
$$z \in S. \tag{4.3.21}$$

Thus

$$\begin{aligned} &\chi_+^{-1/2}(z^{(1)}) \log \xi_+(z) - X_-^{-1/2}(z^{(1)}) \log \xi_-(z) \\ &\quad = X_+^{-1/2}(z^{(1)}) \log \qquad \text{(right-hand side of (4.3.21))} \end{aligned} \tag{4.3.22}$$

Using the Plemelj formula [29], and allowing for the multi-valued nature of the log function, we find [31]

$$X^{-1/2}(z^{(1)}) \log \xi(z) = \psi(z) \tag{4.3.23}$$

where

$$\psi(z) = -(2\pi i)^{-1} \int_S dt\,(t-z)^{-1} X_+^{-1/2}(t^{(1)}) \left\{ \log \rho(t) + \sum_{j=1}^{\nu} \log(t-\alpha_j) \right.$$
$$\left. - \sum_{j=\nu+1}^{l-1} \log(t-\alpha_j) + 2(n-\nu)\log(t-b_1) \right\}$$
$$- \sum_{j=1}^{2(l-1)} \eta_j \int_{L_j} dt\,(t-z)^{-1} X_+^{-1/2}(t^{(1)}). \tag{4.3.24}$$

Here L_j is an arc, not intersecting S, running from b_1 to b_{j+1}, and η_j is integer. This will be a solution provided $\psi(z) \sim z^{-l}$ as $z \to \infty$, which leads after a little manipulation to the conditions

$$\sum_{j=1}^{\nu} \int_{\infty^{(1)}}^{\alpha_j^{(2)}} dw_k + \sum_{j=\nu+1}^{l-1} \int_{\infty^{(1)}}^{\alpha_j^{(1)}} dw_k = -(\pi i)^{-1} \int_S dz\, X_+^{-1/2}(z^{(1)})\, z^{k-1} \log \rho(z)$$
$$- 2n \int_{\infty^{(1)}}^{b_1} dw_k + \sum_{j=1}^{2(l-1)} \eta_j \Omega_{k_j},$$
$$k = 1,\dots, l-1. \tag{4.3.25}$$

In the first set of integrals on the left the contour must cross S once only near b_1, and in the second set on the left the contour must not cross S.

The solution for $\chi(z)$ is obtained from (4.3.20) and (4.3.23),

$$\chi(z) = (z-b_1)^{n-\nu} \left(\sum_{j=1}^{\nu} (z-\alpha_j) \right) \exp[X^{1/2}(z^{(1)})\, \psi(z)]. \tag{4.3.26}$$

Now it is of course clear from the identification of the zeros of $\chi(z)$, $R^*(z)$ that the two divisors $\alpha_1^{(2)},\dots, \alpha_\nu^{(2)}$, $\alpha_{\nu+1}^{(1)},\dots, \alpha_{l-1}^{(1)}$ and $\bar{c}_1,\dots, \bar{c}_{l-1}$ are the same. Indeed, if the expression (4.3.2) for $\rho(z)$ is substituted into (4.3.26), this equation becomes (3.1.6) relating the zeros and poles of the meromorphic function $\bar{\chi}_2$ of (4.3.8).

In the derivation of (4.3.26) we have not used the fact that $\rho^{-1}(z)$ is a polynomial, so that the conjecture predicts (4.3.26) as the asymptotic form of $\chi_2(z)$ for a wider class of functions using this Riemann surface in terms of $\rho(z)$,

$$\rho(z) = X_+^{1/2}(z^{(1)})(f_{2+}(z^{(1)}) - f_{2-}(z^{(1)})), \qquad z \in S \tag{4.3.27}$$

We can make the same remarks about the prediction of normality as at the end of Section 4.1.

4.3.2. *Bernstein–Szegö Polynomials.* These polynomials, a special case of Akhiezer polynomials, were used in the proof of the results described in Section 1.1. We take $l=1$ and Riemann surface

$$y^2-(z^2-1)=0 \tag{4.3.28}$$

with branch points at $z=\pm 1$. The genus is zero and there is no need for any points $\bar{c}_j$ in (4.3.5), etc. In this case S becomes L, the line segment joining ± 1.

Szegö gives an explicit form for $\bar{\chi}_2(z)$ by mapping $\mathscr{R}$ onto the complex t-plane through

$$z=\tfrac{1}{2}(t+t^{-1}). \tag{4.3.29}$$

We take

$$\begin{aligned} \mathscr{R}_0 &= \text{sheet } 1=\{t\in\mathbb{C}:|t|<1\}\\ \mathscr{R}_2 &= \text{sheet } 2=\{t\in\mathbb{C}:|t|>1\}\\ s &= \{t\in\mathbb{C}:|t|=1\}\end{aligned} \tag{4.3.30}$$

and it is easily checked that

$$\exp(\phi(z))=t, \tag{4.3.31}$$

for this function has a zero at $\infty^{(1)}$ and a pole at $\infty^{(2)}$ as required. For each $a_j\in\mathbb{C}$ there exists a unique β_j with $|\beta_j|>1$ such that

$$a_j=\tfrac{1}{2}(\beta_j+\beta_j^{-1}) \tag{4.3.32}$$

and we have the correspondence

$$\begin{aligned} &a_j^{(1)}:\beta_j^{-1}\\ &a_j^{(2)}:\beta_j. \end{aligned} \tag{4.3.33}$$

From the description (4.3.8) we see that $\bar{\chi}_2(z)$ may be written

$$\bar{\chi}_2(z)=t^n h(t^{-1}), \qquad n\geqslant\lambda \tag{4.3.34}$$

where

$$h(t)=\prod_{j=1}^{\lambda}\,[2\beta_j]^{-1/2}\,(t-\beta_j)]. \tag{4.3.35}$$

The asymptotic forms of $p_j(z)$ for $z\notin S$ (unique for $n\geqslant\lambda$) are

$$\left.\begin{aligned} p_1(z) &\underset{n\to\infty}{\sim} -f_2(z^{(1)})\,t^n h(t^{-1})\\ p_2(z) &\underset{n\to\infty}{\sim} t^n h(t^{-1}) \end{aligned}\right.\qquad |t|>1 \tag{4.3.36}$$

where in this case

$$f_2(z) = \text{const.} + (2\pi i)^{-1} \int_{-1}^{1} d\bar{z}\, (\bar{z}^2 - 1)^{-1/2}\, \rho(\bar{z})(\bar{z} - z)^{-1}, \qquad z \in \mathscr{R}_0 \tag{4.3.37}$$

with $\rho(z)$ given by (4.3.2).

We note that (4.3.20) and (4.3.21) may be combined to give in the present case

$$\bar{\chi}_{2+}(z)\, \bar{\chi}_{2-}(z) = \rho^{-1}(z), \qquad z \in S \tag{4.3.38}$$

since, if (4.3.34) holds for $\bar{\chi}_{2+}(z)$ then

$$\bar{\chi}_{2-}(z) = t^{-n} h(t) \tag{4.3.39}$$

and it may be seen that

$$h(t)\, h(t^{-1}) = \rho^{-1}(z). \tag{4.3.40}$$

The equivalent of (4.3.23), (4.3.24) is

$$h(t^{-1}) = \exp\left\{-(2\pi i)^{-1}(z^2 - 1)^{1/2} \int_{-1}^{1} dz'(z' - z)^{-1}(z'^2 - 1)^{-1/2} \log \rho(z')\right\} \tag{4.3.41}$$

a formula that may be checked directly by changing from z' to t' via (4.3.29).

4.3.3. *Jacobi–Dumas Polynomials.* Jacobi [22] began and Dumas [14] completed the explicit calculation of the continued fraction and diagonal Padé approximants to the square root of a quartic polynomial and a closely related function. The significance of Dumas' result was overlooked for about 65 years. A similar function appears if we study the Akhiezer polynomial for $l = 2$, $\lambda = 0$. In this section we explain how to do the equivalent of Dumas' calculation in this case.

With $l = 2$, the polynomial $X(z)$ of (3.4.4) is of fourth degree and the genus of $\mathscr{R}$ is 1. The only integral of the first kind is the elliptic integral (see Siegel [41, 42])

$$u = \int_{b_1}^{z} dz'\, X^{-1/2}(z'). \tag{4.3.42}$$

We can map $\mathscr{R}$ onto the complex u-plane with (4.3.47). In this way the u-plane consists of many copies of $\mathscr{R}$, as all points of the form $u = 2m\omega +$

$2m'\omega'$, m, m' integer, correspond to the same point in $\mathscr{R}$. By ω, ω' we mean the half periods of (4.3.42),

$$\omega = \int_{b_1}^{b_2} dz'\, X^{-1/2}(z') \tag{4.3.43}$$

$$\omega' = \int_{b_2}^{b_3} dz'\, X^{-1/2}(z'). \tag{4.3.44}$$

Let us define v by

$$v = \int_{b_1}^{\infty} dz'\, X^{-1/2}(z'). \tag{4.3.45}$$

Then the points $z \in \mathscr{R} = \infty^{(1)}, \infty^{(2)}, b_1, b_2, b_3, b_4$ correspond to $u = v, -v, 0, \omega, \omega + \omega', \omega'$, respectively.

In this case $\bar{\chi}_2(z)$ has

$$\begin{aligned} \bar{\chi}_2(z)\!:\quad &\text{zeros: } (\infty^{(1)})^{n-1}, \bar{c}_1 \\ &\text{poles: } (\infty^{(2)})^{n}. \end{aligned} \tag{4.3.46}$$

Abel's theorem (3.1.6) shows that, for these points to be the poles and zeros of a meromorphic function, the sum of the u-values corresponding to the zeros must equal the sum for the poles, up to integer multiples of the periods. Thus, if

$$\gamma_n = \int_{b_1}^{\bar{c}_1} dz'\, X^{-1/2}(z') \tag{4.3.47}$$

then

$$\gamma_n + (n-1)v = -nv$$

so that

$$\gamma_n = -(2n-1)v \tag{4.3.48}$$

is one value of u that corresponds to $\bar{c}_1$. Because the genus is 1, $\bar{c}_1$ is always unique, as are the polynomials $p_1(z), p_2(z)$.

We may now construct $\bar{\chi}_2(z)$ in terms of the Weierstrass function $\sigma(u)$, an entire function with zero only at the origin and points shifted from there by multiples of the periods. It satisfies [44]

$$\begin{aligned} &\sigma(u + 2M\omega + 2N\omega') \\ &\quad = (-1)^{M+N+MN}\,\sigma(u)\exp[(u + M\omega + N\omega')(2M\eta + 2N\eta')] \end{aligned} \tag{4.3.49}$$

where

$$\eta = \zeta(\omega), \eta' = \zeta(\omega') \tag{4.3.50}$$

ζ being the Weierstrass ζ-function [44]. We have

$$\tilde{\chi}_2(z) = \frac{\sigma(u-\gamma_n)[\sigma(u-v)]^{n-1}}{[\sigma(u+v)]^n} \tag{4.3.51}$$

for this function is doubly periodic (from (4.3.49)) and has zeros, poles at the appropriate places.

Following Dumas [14] we set

$$\tilde{\chi}_2(z) = h_n(u)[k(u)]^n \tag{4.3.52}$$

where, with $v = pw + p'w'$, p, p' real,

$$k(u) = \frac{\sigma(u-v)}{\sigma(u+v)} \exp[u(2p\eta + 2p'\eta')]. \tag{4.3.53}$$

It may be shown, with the help of (4.3.49), that

$$k(u) = \exp[\phi(z)] \tag{4.3.54}$$

with $\phi(z)$ given by (3.4.5), and that $|h_n(u)|$ is bounded.

Points on sheet 2, $\mathscr{R}_2$, correspond to those u for which $|k(u)| > 1$. The corresponding point on sheet 1, $\mathscr{R}_0$, is obtained from $-u$, and in the u-plane the curve s corresponds to $|k(u)| = 1$, called Σ by Dumas [14]. In the z-plane the equivalent set S consists of two analytic arcs ending at the branch points $b_j, j = 1,..., 4$.

The polynomial $p_2(z)$ is given exactly by

$$p_2(z) = h_n(u)[k(u)]^n + h_n(-u)[k(-u)]^n. \tag{4.4.55}$$

We can see from this explicit form the already known fact that

$$p_2(z) \underset{n\to\infty}{\sim} \tilde{\chi}_2(z), \qquad z \notin S \tag{4.3.56}$$

where u is chosen so that $|k(u)| > 1$. If γ_n corresponds to $\bar{c}_1 \in \mathscr{R}_2$ then $\tilde{\chi}_2(z)$ will be zero at $\bar{c}_1$ and $\tilde{p}_2(z)$ will be zero near $\bar{c}_1$. From (1.2.10) the error in the $[n/n]$ Padé approximant will be small, $z \notin S$, except near to $\bar{c}_1$ if $\bar{c}_1 \in \mathscr{R}_2$.

The variation of $\bar{c}_1 \in \mathscr{R}$ with n will appear irregular if v is incommensurate with ω, ω', although in the u-plane the corresponding point γ_n is linear in n. We may solve (4.3.47) to obtain the explicit form

$$\bar{c}_1 = \zeta(\gamma_n + v) - \zeta(\gamma_n - v) - 2\zeta(v) + b_1. \tag{4.3.57}$$

With the help of the form of $p_2(z)$ it is possible to work out the coefficients in the three-term recurrence relation connecting polynomials of adjacent degrees, which were first given by Jacobi [22].

4.3.4. *Examples with $m > 2$.* Let us return to the situation of Section 4.1 but remove the restrictions on the surface $\mathscr{R}$. If we suppose that $\lambda = m - 1 + g$, then $\bar{R}(z)$ is easy to construct, for the additional zeros $\{\bar{c}_j\}$ are g in number and may be obtained as described in Section 3.1. The type I polynomials follow immediately.

For example [35], let us take $m = 3$ and choose $\mathscr{R}$ to be

$$\mathscr{R}: zy^3 = z - 1. \tag{4.3.58}$$

This surface, of genus 0, has branch points at $z = 0, 1$. Because $g = 0$, $\exp(\phi(z))$ is meromorphic with

$$\exp(\phi(z)):\quad \begin{aligned} &\text{zeros: } (\infty^{(1)})^2 \\ &\text{poles: } \infty^{(2)}, \infty^{(3)} \end{aligned} \tag{4.3.59}$$

so that

$$\exp(\phi(z)) = z(1 - y)^3. \tag{4.3.60}$$

The boundaries of adjacent sheets chosen according to the prescription of Section 3.1 all correspond to real z. They are

$$\begin{aligned} S' &= \{\text{boundary between sheets } 1, 2\} = \{z: 0 \leqslant z \leqslant 1\} \\ S &= \{\text{boundary between sheets } 2, 3\} = \{z: z \leqslant 0 \text{ or } 1 \leqslant z\}. \end{aligned} \tag{4.3.61}$$

Now suppose we choose $f_1(z) = 1$, $f_2(z) = y$, $f_3(z) = y^2$, an example discussed by Shafer [40], so that $f_2(z)$ has a simple pole at the branch point $z = 0$, and $f_3(z)$ a double pole there. In the notation of Section 4.1, $a_1 = a_2 = 0$. The function $\bar{R}(z)$ has

$$\bar{R}(z):\quad \begin{aligned} &\text{zeros: } (\infty^{(1)})^{2(n+1)} \\ &\text{poles: } (\infty^{(2)})^n, (\infty^{(3)})^n, (0)^2 \end{aligned} \tag{4.3.62}$$

and so $\bar{R}(z)$ may be written

$$\bar{R}(z) = 3 \exp(n\phi(z))(1 - y)^2 \tag{4.3.63}$$

The polynomials $\{p_j(z)\}$ follow from (4.1.3)–(4.1.5) and are easily determined explicitly in this case. Thus,

$$\begin{aligned}
p_1(z) &= z^n((1-y_1)^{3n+2} + (1-\omega y_1)^{3n+2} + (1-\omega^2 y_1)^{3n+2}) \\
p_2(z) &= z^n((1-y_1)^{3n+2} + \omega^2(1-\omega y_1)^{3n+2} + \omega(1-\omega^2 y_1)^{3n+2})\, y_1^{-1} \\
p_3(z) &= z^n((1-y_1)^{3n+2} + \omega(1-\omega y_1)^{3n+2} + \omega^2(1-\omega^2 y_1)^{3n+2})\, y_1^{-2}
\end{aligned} \tag{4.3.64}$$

where y_1 corresponds to sheet 1 and $\omega = \exp(2\pi i/3)$.

For type II polynomials similar constructions are possible. Thus, from (4.2.5), $\bar{T}(z)$ may have poles only at the poles of $\{H_j(z)\}$ and at $(\infty^{(1)})^{(m-1)n}$. If the set of poles of $\{H_j(z)\}$ is $m-1+g$ in number, then, after the zeros at $(\infty^{(2)})^{n+1}, \ldots, (\infty^{(m)})^{n+1}$, g zeros of $\bar{T}(z)$ remain, which may be determined as in Section 3.1.

Thus, in the example above, $H_j(z)$, $j = 1, 2, 3$, are proportional to $1, y^{-1}, y^{-2}$, so that the set of their poles is $(1)^2$ and $\bar{T}(z)$ has

$$\bar{T}(z): \quad \begin{array}{l} \text{zeros: } (\infty^{(2)})^{n+1}, (\infty^{(3)})^{n+1} \\ \text{poles: } (\infty^{(1)})^{2n}, (1)^2 \end{array} \tag{4.3.65}$$

which gives

$$\bar{T}(z) = \exp(-n\phi(z))(y^2 + y + 1)\, y^{-2}. \tag{4.3.66}$$

From (4.2.11) we find, with $h(y) = 1 + y + y^2$,

$$\begin{aligned}
q_1(z) &= z^{2n}((h(y_1))^{3n+1} + (h(\omega y_1))^{3n+1}\,\omega + (h(\omega^2 y_1))^{3n+1}\,\omega^2)\, y_1^{-2} \\
q_2(z) &= z^{2n}(h(y_1)^{3n+1} + (h(\omega y_1))^{3n+1}\,\omega^2 + (h(\omega^2 y_1))^{3n+1}\,\omega)\, y_1^{-1} \\
q_3(z) &= z^{2n}(h(y_1)^{3n+1} + (h(\omega y_1))^{3n+1} + (h(\omega^2 y_1))^{3n+1}).
\end{aligned} \tag{4.3.67}$$

These examples give asymptotic forms which satisfy the equations of the conjecture, and other cases are easily constructed for which the conjecture may also be tested.

4.4. *Rigorous Generalizations of the Results of Section* 4.3

The results of Bernstein–Szegö [45], referred to in the introduction, relate to the surface $y^2 = z^2 - 1$ of Section 4.3.2, with functions $f_1(z) = 1$ and $f_2(z)$ such that it can be approximated adequately by the form (4.3.43). More precisely, it is required that

$$f_2(z) = \text{const.} + (2\pi i)^{-1} \int_L d\bar{z}\, X_+^{-1/2}(\bar{z}^{(1)})\, \sigma(\bar{z})(\bar{z} - z)^{-1}, \qquad z \in \mathscr{R}_0 \tag{4.4.1}$$

with $X(z) = z^2 - 1$ and $\sigma(z)$ a strictly positive function, $z \in S$, satisfying the smoothness condition (1.1.8). The results for real $\sigma(z)$ were generalized to complex $\sigma(z)$ first by Baxter [5, 6] and later by Nuttall [37, 32].

The original proof of Bernstein–Szegö [45], as well as the later work, applied to polynomials orthogonal on the unit circle. From these polynomials can be constructed polynomials orthogonal on L with the help of the transformation (4.3.29). An argument in [32], repeated from [37], completes the discussion by showing that, if the polynomials on the unit circle are unique, as they are for large enough n, then the polynomials on L are also unique.

Both generalizations give the same asymptotic results as those for real weight $\sigma(z)$ described in Section 1.1 and of course these results follow from the conjecture of Section 3.2. The conditions on $\sigma(z)$ are different, however. Baxter requires $\sigma(\cos\theta)$ to be integrable over θ, $-\pi \leqslant \theta \leqslant \pi$ and the Fourier coefficients of $\log \sigma(\cos\theta)$ have to be such that the sum of their absolute values converges. Nuttall requires the same conditions as Bernstein–Szegö except that $\sigma(z)$ may be complex.

It does not appear likely that the method of Baxter can be extended to apply to polynomials orthogonal on sets S such as those of Section 4.3.1, for then the polynomials may have zeros away from S. However, the integral equation method of Bernstein–Szegö [45], used also by Nuttall (37), has been modified by Nuttall and Singh [31] to handle this case. We now summarize the main points of this work.

We consider the surface (3.4.4) and suppose that no zero of $Y(z)$ of (3.4.7) lies on S, so that S consists of l components. We take $f_1(z) = 1$ and set

$$f_2(z) = \text{const.} + (2\pi i)^{-1} \int_S d\bar{z}\, X_+^{-1/2}(\bar{z}^{(1)})\, \sigma(\bar{z})(\bar{z} - z)^{-1}, \qquad z \in \mathscr{R}_0 \tag{4.4.2}$$

where $\sigma(z)$, $z \in S$, is a non-vanishing complex function satisfying smoothness requirements set out in [31] that are analogous to (1.1.8). They lead to the result that $\rho_n(z)$, the inverse of a polynomial of degree $n - l + 1$, can be found such that, with $\lambda > 0$,

$$\sup_{z \in S} |\sigma(z) - \rho_n(z)| < \text{const.}(\log n)^{-1-\lambda}. \tag{4.4.3}$$

Now suppose that polynomials $p_2(z)$ of degree n, $n+1$ corresponding to $f_2(z)$ of (4.4.2) with $\sigma(z)$ replaced by $\rho_n(z)$ are denoted by $p(z)$, $p'(z)$. These are Akhiezer polynomials with properties as described in Section 4.3.1. We take $\alpha = \alpha_1, \ldots, \alpha_{l-1}$ (also called $\bar{c}_1 \cdots \bar{c}_{l-1}$) and $\alpha' = \alpha'_1, \ldots, \alpha'_{l-1}$ to be the divisors corresponding to the two polynomials and construct $\bar{p}(z), \bar{p}'(z)$ from the appropriate $X(z)$ of (4.3.20) after multiplying by the normalizing factor

$$\prod_{j=1}^{l-1} (1+|\alpha_j|)^{-1/2}. \tag{4.4.4}$$

We assume that both the divisors are unique and that no $\alpha_j, j=1,\ldots, l-1$, is on sheet 1 satisfying $|\alpha_j|^{-1} < \varepsilon$ and no $\alpha'_j, j=1,\ldots, l-1$, is on sheet 2 with $|\alpha'_j|^{-1} < \varepsilon$. In [31] we showed that it is possible to find $\varepsilon > 0$ such that, for large enough n, values of n exist, consecutive ones differing by no more than l, that satisfy these conditions (see Section 4.5). In case any $\alpha_j = \infty^{(2)}$, we use the limit as $\alpha_j \to \infty^{(2)}$ to define $\bar{p}(z)$ and similarly for $\bar{p}'(z)$ in case any $\alpha'_j = \infty^{(1)}$. Thus $\bar{p}'(z)$ is always of degree $n+1$ but $\bar{p}(z)$ may be of degree $<n$.

The integral equation is constructed by making use of the properties of the reproducing kernel for weight $X^{-1/2}(z)\,\rho_n(z)$. The general case of Section 2.2 reduces to a form analogous to that used by Bernstein [7]. The equation for $p_2(z)$, the polynomial of degree n corresponding to $f_2(z)$ of (4.4.2) is

$$\begin{aligned} p_2(z) = \mu_n^{-1}\bar{p}(z) + \mu_n^{-1} \int_S d\bar{z}\, X_+^{-1/2}(\bar{z}^{(1)})(\sigma(\bar{z}) - \rho_n(\bar{z}))(\bar{p}(z)\,\bar{p}'(\bar{z}) \\ -\bar{p}'(z)\,\bar{p}(\bar{z}))(\bar{z}-z)^{-1}\,p_2(\bar{z}) \end{aligned} \tag{4.4.5}$$

where μ_n is bounded and non-zero when the conditions on the divisors hold. In this case it was shown in [31] that, no matter what polynomial of degree n $p_2(z)$ may be, the integral on the right of (4.4.5) may be bounded by const. $(\log n)^{-\lambda}$ $\sup_{\bar{z}\in S} |p_2(\bar{z})|$ for $z \in S$. This means that for large n the integral equation may be solved by iteration and we have, for all finite z,

$$p_2(z) = \mu_n^{-1}\bar{p}(z) + \text{const.}(\log n)^{-\lambda} \exp(n \operatorname{Re} \phi(z^{(2)}))\; O(1) \tag{4.4.6}$$

showing that, except near the zeros of $\bar{p}(z)$, the dominant contribution is

$$p_2(z) \underset{n\to\infty}{\sim} \mu_n^{-1}\bar{p}(z). \tag{4.4.7}$$

The asymptotic form of the Akhiezer polynomial $p(z)$ is given in terms of $\chi(z)$ from (4.3.20), with $\rho_n(z)$ substituted for $\rho(z)$ in (4.3.21), (4.3.24). For large n, (4.4.3) shows that this $\chi(z)$ approaches that we would find by using $\sigma(z)$ in place of $\rho(z)$. As we remarked in Section 4.3.1, this $\chi(z)$ is the (usually) unique solution of the conjecture for $f_2(z)$ given by (4.4.2).

We can summarize by stating fact, for functions of the form (4.4.2) with the restrictions on S and $\sigma(z)$ described, the conjecture correctly and uniquely predicts the asymptotic behavior of the polynomials $p_1(z)$, $p_2(z)$

provided that the divisors α, α' obtained by solving (4.3.28) with $\sigma(z)$ replaced by $\rho(z)$ satisfy the conditions

(i) α, α' are unique and not close to being special

(ii) no α_j is near $\infty^{(1)}$, no α_j' is near $\infty^{(2)}$ (see Sec. 4.5).

These conditions are no doubt too restrictive, and the conjecture probably holds in this case for all n. In any case, as was shown in [31], relations between the polynomials included in the above conditions and the intervening ones can be found that lead to the result that the diagonal Padé approximants to $f_2(z)$ of (4.4.2) converge in capacity, outside S.

When $l=1$, we are back to the Bernstein–Szegö case with complex weight. All sufficiently large values of n meet the conditions and the Nuttall–Singh argument above provides an alternative proof of the conjecture in this case.

4.5. *Asymptotic Normality*

In the examples discussed in Section 4.1, 4.2, 4.3 for which $g \geqslant 1$, the normality of diagonal type I polynomials of degree $n-1$ depended asymptotically on a divisor $c(n) = c_1(n) \cdots c_g(n)$. Thus, if we define, for $\varepsilon \geqslant 0$,

Condition $N_\varepsilon(c)$: No c_j, $j=1,..., g$, is within ε of $\infty^{(1)}$ and c is not within ε of any special divisor (4.5.1)

then if $N_\varepsilon(c(n))$ is true for some fixed $\varepsilon > 0$, the corresponding point $\rho_n = \{\rho_j = n,\ j=1,..., n\}$ is normal for large n. Similarly for type II polynomials of degree n there is a divisor $t(n)$ such that, if $N_\varepsilon(t(n))$ is true, ρ_n is normal. The equation for $c(n)$, for example (4.1.10) or (4.3.9), may be put in the form

$$c(n), (\infty^{(1)})^{(m-1)n} \sim (\infty^{(2)})^n,..., (\infty^{(m)})^n, d \tag{4.5.2}$$

where the symbol $\sim$ means that a meromorphic function exists with zeros, poles at points on the left, right of (4.5.2), respectively. Here, d is a fixed divisor of g points. There is a similar relation for $t(n)$, but it turns out using (4.2.12), that $c(n)$, $t(n)$ are connected by

$$c(n), t(n),\ (\infty^{(2)})^2,..., (\infty^{(m)})^2 \sim b_1,..., b_\nu. \tag{4.5.3}$$

Note that the branch points $\{b_j\}$ have been assumed to all have winding number 2, but this restriction could probably be easily removed in this section.

In [31] it was proved that, in the case $m=2$, it is possible to find $\varepsilon > 0$ and n_0 such that the length of the longest sequence of consecutive integers with $n > n_0$ for which $N_\varepsilon(c(n))$ is not true is at most $g+1$. We suspect that a similar result is true in general for $m > 2$. For example, consider the case $g=1$, $m=3$. If $c_1(n) = c_1(n+1) = \infty^{(1)}$, then it follows from (4.5.2) that

$$\infty^{(2)}\infty^{(3)} \sim (\infty^{(1)})^2 \tag{4.5.3}$$

which is equivalent to the statement that $\exp(\phi(z))$ is meromorphic, that is, has only zero periods. If $\mathscr{R}$ does not possess this property, then we could no doubt show that no two consecutive integers exist such that $N_\varepsilon(c(n))$ is not true in this example. The general case remains to be discussed.

Another interesting point concerns normality expressed in terms of type I or type II polynomials. As we remarked in Section 1.3, Loxton and van der Poorten [26] showed that, if the type I polynomials at ρ_n are unique and the remainder is of lowest possible order, then the corresponding type II polynomials are unique and $Q_1(\rho_n, 0) \neq 0$. In view of our relation between normality and the condition $N_\varepsilon(c)$, we expect that the following should hold. If $c(n)$, $t(n)$ are related by (4.5.3), then $N_0(c(n))$ true implies $N_0(t(n))$ true and vice versa.

For the case $m=2$, this statement is easy to prove, using the form of special divisors given in [31]. If $g=1$, $m>2$, there can be no special divisors and again the statement follows from a consideration of the meromorphic function $W(z)$ defined by

$$W(z)\,dz = dw_1 \tag{4.5.4}$$

where dw_1 is the differential of the first kind (see Section 3.1). The function $W(z)$ has first order poles at $b_1,\ldots, b_\nu$ and second order zeros at $\infty^{(1)},\ldots, \infty^{(m)}$. We expect that an extension of this argument might prove the statement true in general.

4.6. *Hypergeometric Functions*

Chudnovsky [12] has constructed a number of examples involving hypergeometric functions and their generalizations for which the H–P polynomials can be determined exactly. We discuss one such case in this section and another in Section 4.7.

Here we take

$$\begin{aligned} F_1(x) &= 1 \\ F_j(x) &= {}_2F_1(1, \omega_j; c; x), \qquad j = 2,\ldots, m \end{aligned} \tag{4.6.1}$$

where no two of 1, $\{\omega_j\}$, c differ by an integer. Chudnovsky [12] was able to generalize the work of Mahler [27] (who worked with $c=1$) to give an explicit form for the remainder for type I polynomials defined by (1.3.1). Thus in this case (with $\omega_1 = 0$)

$$\sum_{j=1}^{m} P_j(\rho, x) F_j(x) = R(x) \tag{4.6.2}$$

where

$$R(x) = \text{const.} \int ds\, {}_2F_1(1, s; c; x) \left[\prod_{j=1}^{m} Z_j(s, \omega_j) \right]^{-1} \tag{4.6.3}$$

with

$$Z_j(s, \omega) = \prod_{k=0}^{\rho_j} (s + k - \omega). \tag{4.6.4}$$

(Note that for this section only it is convenient to have a somewhat different definition of the function $R(x)$.)

We [4] have shown, after Chudnovsky [12], how to obtain from (4.6.3) the asymptotic behavior of $R(x)$, and from that $P_j(\rho, x)$, in the diagonal case. The results are consistent with (3.2.4) and (3.2.5) for an appropriate Riemann surface. Our main aim here is to work out the behavior of type II polynomials by using their connection (2.1.3) with type I polynomials, and we shall not describe the derivation of the required type I asymptotics. A technical problem makes the calculation for diagonal type II polynomials somewhat involved, so that we shall instead treat a near-diagonal example.

From now on we restrict attention to the case $m = 3$ and suppose that $\{\rho_i\}$ to be used in (4.6.2) have the form $\rho_i = n + 1 + \sigma_i$, $i = 1, 2, 3$, with σ_i independent of n. The technique of [4] leads, after an appropriate choice of constant in (4.6.3), to

$$\begin{aligned} R(x) = {} & (1-y)^{3n+1+c+\sigma_1+\sigma_2+\sigma_3}(-y)^{-\omega_2-\omega_3-c}(1-z)^{c-1} \\ & \times \{1 + n^{-1}[C + \sigma_1 - 6^{-1}(1-y)y((\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 \\ & + (\sigma_1-\sigma_2)(\sigma_2-\sigma_3))] + O(n^{-2}) \end{aligned} \tag{4.6.5}$$

where, as always $z = x^{-1}$, and

$$y = (1-x)^{1/3}. \tag{4.6.6}$$

The constant C in (4.6.5) does not depend on the values of σ_i. The value of y to be used in (4.6.5) when $x \approx 0$ is $y \approx 1 - x/3$.

In this case the surface of the conjecture turns out to be

$$\mathscr{R}\colon y^3 = 1 - x = 1 - z^{-1} \tag{4.6.7}$$

as in Section 4.3.4. The three points $x^{(1)}, x^{(2)}, x^{(3)} \in \mathscr{R}$ corresponding to the same point in the x-plane are such that

$$|1-y_3| \geqslant |1-y_2| \geqslant |1-y_1| \qquad (4.6.8)$$

where y_j means $y(x^{(j)})$. Thus $\mathscr{R}$ may be mapped on to the y-plane as follows

$$\begin{aligned} &\text{Sheet 1: } |\operatorname{Arg} y| < \pi/3 \\ &\text{Sheet 2: } \pi/3 < |\operatorname{Arg} y| < 2\pi/3 \\ &\text{Sheet 3: } 2\pi/3 < |\operatorname{Arg} y| \leqslant \pi. \end{aligned} \qquad (4.6.9)$$

The form (4.6.5) is initially valid in sheet 1 and we presume that it may be continued to represent the asymptotic behavior of the continuation of $R(x)$ along a path from $x^{(1)}$ to $x^{(2)}$ to $x^{(3)}$ such that $|1-y|$ is increasing, provided we avoid branch points. Thus if $\operatorname{Arg} y_1 > 0$, we may follow a circle in the y-plane clockwise to $y_2 = y_1 \exp(-2\pi i/3)$ to $y_3 = y_1 \exp(-4\pi i/3)$. If $\arg y_1 < 0$ the circle must be followed anti-clockwise. Of course (4.6.5) is not valid near its singularities.

Following our previous approach, the polynomials $P_j(\rho, x)$ are obtained by solving

$$\sum_{j=1}^{3} P_j(\rho, x)\, F_j(x^{(i)}) = R(x^{(i)}), \qquad i = 1, 2, 3 \qquad (4.6.10)$$

where the continuations in $F_j(x)$, $R(x)$ are made as explained above. In [4] we treated diagonal type I polynomials but now we choose sets $\sigma_i^{(k)}$, i, $k = 1, 2, 3$, so that $\mu_i^{(1)}$ of (2.1.2) is given by $\mu_1^{(1)} = \mu_3^{(1)} = n$, $\mu_2^{(1)} = n+1$. This means that

$$\sigma_i^{(k)} = \delta_{1i} + \delta_{ik} + \delta_{2i} - 1, \qquad i, k = 1, 2, 3 \qquad (4.6.11)$$

makes $\rho_i^{(k)} = n + 1 + \sigma_i^{(k)}$ agree with (2.1.1). Solving (4.6.10) for the various $P_j(\rho^{(k)}, x)$ and then solving (2.1.3) gives

$$Q_j(\mu^{(1)}, x) = \sum_{k=1}^{3} F_j(x^{(k)})\, T_k(x) \qquad (4.6.12)$$

where

$$T_k(x) = x^{3n+2} (\det R)^{-1} \sum_{i,j=1}^{3} \varepsilon_{ijk} R^{(2)}(x^{(i)})\, R^{(3)}(x^{(j)}). \qquad (4.6.13)$$

We have written $R^{(k)}(x)$ as the remainder in (4.6.2) corresponding to $\rho^{(k)}$, and

$$\det R = \det[R^{(i)}(x^{(j)})], \qquad i, j = 1, 2, 3. \qquad (4.6.14)$$

From (4.6.10) we have

$$\det R = \det P \det F$$
$$= x^{3n+2} \det[F_i(x^{(j)})], \qquad i, j = 1, 2, 3 \tag{4.6.15}$$

from (2.1.6).

Now suppose that $x \notin S'$, where S' is the curve on which $|1 - y_1| = |1 - y_2|$, or in this case $\{x\colon \operatorname{Im} x = 0,\ x > 1\}$. We see from (4.6.5), (4.6.13) that $|T_1(x)| \gg |T_2(x)|, |T_3(x)|$ so that, as $n \to \infty$,

$$Q_j(\mu^{(1)}, x) \sim F_j(x^{(1)})\, T_1(x). \tag{4.6.16}$$

Substituting (4.6.5) in (4.6.13) then shows that, up to a constant multiple,

$$T_1(x) \sim [\det F]^{-1}\, [(1 - y_2)(1 - y_3)]^{3n+1+c}\, (y_2 y_3)^{-\omega_2 - \omega_3 - c}$$
$$\times (1 - z^{(2)})^{c-1}\, (1 - z^{(3)})^{c-1}\, (y_2 - y_3)(1 - y_2 - y_3). \tag{4.6.17}$$

If we had chosen $\mu_i^{(1)} = n$, $i = 1, 2, 3$, the contribution to (4.6.17) coming from the terms of order n^{-1} would have vanished and it would be necessary to work out the $O(n^{-2})$ term in (4.6.5).

The work of [4] shows that

$$\det F = \text{const.}(1 - z)^{2c - \omega_2 - \omega_3 - 2} z^{\omega_2 + \omega_3}. \tag{4.6.18}$$

Now it might appear that (4.6.17) is discontinuous as y_1 crosses the positive real axis on account of the fact that y_2, y_3 change discontinuously at this point, but a careful check shows, with the help of (4.6.18) that (4.6.17) is in fact continuous and $T_1(x)$ can be represented by

$$T_1(x) \sim T(z) = [x/(1 - y)]^{3n+1+c}\, y^{\omega_2 + \omega_3 - 2c + 1}(1 + y), \tag{4.6.19}$$

which is analytic in $|\arg y| < 2\pi/3$, i.e., in $\mathscr{R}_0$ as the conjecture requires.

Let us now consider the construction of $\psi(z)$, analytic in $\mathscr{R}_3(|\arg y| > 2\pi/3)$, which should be related to $T(z)$ by (3.2.13). We choose $g_1 = 1$, $g_2 = y$, $g_3 = \mu y^2$, so that $G(z)$ of (3.2.8) turns out to be

$$G(z) = \text{const.}(1 - x), \tag{4.6.20}$$

and pick the constant μ to make

$$G(z) = 1 - x, \qquad \arg y = 2\pi/3. \tag{4.6.21}$$

On the other part of s, $\arg y = -2\pi/3$, we will have

$$G(z) = -(1 - x), \qquad \arg y = -2\pi/3. \tag{4.6.22}$$

We assert that $\psi(z)$ may be written as

$$\psi(z) = \lambda[x/(1-y)]^{3n+1+c}\,(1-z)^{2c-\omega_2-\omega_3-2}\,z^{\omega_2+\omega_3}y^{\omega_2+\omega_3-2c+1}$$
$$\times(1+y)(1-x)^{-1}, \qquad (4.6.23)$$
$$z \in \mathscr{R}_3 = \{y\colon |\arg y| > 2\pi/3\}$$

where the constant λ is chosen appropriately. It must be checked that (3.2.13) holds on $s = \{y\colon |\arg y| = 2\pi/3\}$. In doing this the reader must remember the definition of $D(z)$ to be used. Four different values of the constant in (4.6.18) are required, corresponding to $\arg y = \pm 2\pi/3$, $|y| > 1$, <1.

We have shown that

$$Q_j(\mu^{(1)}, x) \underset{n\to\infty}{\sim} F_j(x^{(1)})\,T(z^{(1)}), \qquad j = 1, 2, 3 \qquad (4.6.24)$$

for $x \notin S'$, i.e., $|\arg y| < \pi/3$, where $T(z)$ solves the boundary value problem of the conjecture. In this case the points on sheets 2, 3 corresponding to both $z = 0$, ∞ lie on s, and the functions $F_j(x)$ are singular there. This complicates the specification of further conditions on the solution of the boundary value problem needed to complete the conjecture, and the question of how to state these conditions in a case such as this remains unanswered.

4.7. *Contiguous Generalized Hypergeometric Functions*

We are concerned here with finding H–P polynomials in the case when

$$F_1(x) = {}_mF_{m-1}(a_1,\ldots, a_m; c_2,\ldots, c_m; x) \qquad (4.7.1)$$

and $F_j(x)$, $j = 2,\ldots; m$ are functions contiguous to these, which means that they are of the same form with parameters $\{a_k\}$, $\{c_k\}$ differing by integers from those of (4.7.1). All such contiguous functions lie in a Riemann module, as we shall explain below, and Chudnovsky [12] used his general theory to construct the polynomials of type I for a particular choice of $F_j(x)$, $j = 1,\ldots, m$. Unfortunately, the general theory does not apply to this case (unless $m = 2$) because two or more of the exponents of the solutions valid near $x = 1$ of the mth order differential equation satisfied by (4.7.1) differ by integers so that Theorem 4.5 of (12) is incorrect.

In this section we extend the ideas of Chudnovsky by introducing the concept of the dual Riemann module and compute its form in the present case. This incidentally leads to relations between generalized hypergeometric functions which may be new. For an example with $m = 3$ we find an expression for both types of H–P polynomials, those of type II involving the dual module. Because of the explicit nature of our solution we can determine the asymptotic form of the polynomials, and it is found that we have an example of case 2 of the conjecture.

The method of the example is easily extended to the non-diagonal case, other choices of contiguous $F_j(x)$ and higher values of m. Having found the H–P polynomials we could easily determine the coefficients appearing in the formulae relating sets of neighboring degrees and in this way obtain a generalization of the Gauss continued fraction for ${}_2F_1(a_1+1,\ a_2+1;\ c_2+1; x/{}_2F_1(a_1, a_2; c_2; x)$.

4.7.1. *Dual Module.* Throughout we suppose that no pair of $a_1,..., a_m$, $c_1,..., c_m$, where $c_1 = 1$, differ by an integer. Let

$$y_1^{(0)}(x) = {}_mF_{m-1}(a_1,..., a_m;\ c_2,..., c_m; x) \tag{4.7.2}$$

where the generalized hypergeometric function is given by a power series [12] convergent for $|x| < 1$. This function satisfies the differential equation of order m

$$\left\{\prod_{t=1}^{m} (\theta + a_t) - x^{-1} \prod_{t=1}^{m} (\theta + c_t - 1)\right\} y = 0 \tag{4.7.3}$$

with $\theta = xd/dx$. Smith [43] has given two sets of m independent solutions of (4.7.3), which are

$$\begin{aligned} y_j^{(0)}(x) = x^{1-c_j}\, {}_mF_{m-1}(1 + a_1 - c_j,..., 1 + a_m - c_j;\ 1 + c_1 - c_j,...,\\ [1 + c_j - c_j],..., 1 + c_m - c_j; x), \qquad j = 1,..., m \end{aligned} \tag{4.7.4}$$

and

$$\begin{aligned} y_j^{(\infty)}(x) = x^{-a_j}\, {}_mF_{m-1}(1 - c_1 + a_j,..., 1 - c_m + a_j;\ 1 - a_1 + a_j;...,\\ [1 - a_j + a_j],..., 1 - a_m + a_j; x^{-1}), \qquad j = 1,..., m \end{aligned} \tag{4.7.5}$$

where the parameter in [ ] is omitted. We take all these functions to be single-valued in the x-plane cut from 0 to ∞ along the real axis.

Let us represent these solutions in the form $\underline{Y}^{(0)}$, $\underline{Y}^{(\infty)}$, m-dimensional column matrices. Smith [43] has shown that

$$\underline{Y}^{(0)} = H\underline{Y}^{(\infty)} \tag{4.7.6}$$

where the $m \times m$ matrix H has the form

$$H = C^{-1}GA \tag{4.7.7}$$

with

$$\begin{aligned} C &= \mathrm{diag}(\exp(i\pi c_j)) \\ A &= \mathrm{diag}(\exp(i\pi a_j)) \end{aligned} \qquad j = 1,\ldots, m \tag{4.7.8}$$

and

$$G_{jk} = \frac{\Gamma(c_j - a_k)\,\Gamma(1 + c_k - c_j)}{\Gamma(c_k - a_k)} \prod_{t \neq k}^{m} \frac{\Gamma(a_t - a_k)\,\Gamma(1 + c_t - c_j)}{\Gamma(c_t - a_k)\,\Gamma(1 + a_t - c_j)}. \tag{4.7.9}$$

Smith also showed that $(G^{-1})_{jk}$ is given by the above formula with $c \to -a$, $a \to -c$.

Now the equation (4.7.3) and all its solutions, or their analytic continuations, are singular at $x = 0, 1, \infty$ and nowhere else. The monodromy matrix for a given singular point relates m independent solutions of (4.7.3) to their continuations taken along a small circular path around the singular point. The particular form of a monodromy matrix depends on the choice of m independent functions. We shall use $y_1^{(0)}(x)$ and its continuations round $x = \infty$ $1, 2,\ldots, m-1$ times, and these functions, analytic in the cut x-plane, will be denoted by $W_j(x)$, $j = 1,\ldots, m$. Thus

$$\begin{aligned} W_1(x) &= y_1^{(0)}(x) = \underline{H}_1^T \underline{Y}^{(\infty)} = \underline{H}_1^T H^{-1} \underline{Y}^{(0)} \\ W_2(x) &= \underline{H}_1^T A^2 \underline{Y}^{(\infty)} = \underline{H}_1^T A^2 H^{-1} \underline{Y}^{(0)} \end{aligned} \tag{4.7.10}$$

and so

$$W_j(x) = \underline{H}_1^T A^{2(j-1)} H^{-1} \underline{Y}^{(0)}, \qquad j = 1,\ldots, m \tag{4.7.11}$$

where the column matrix $\underline{H}_1$ is given by

$$(\underline{H}_1)_j = H_{1j}, \qquad j = 1,\ldots, m. \tag{4.7.12}$$

We have made use of the special form (4.7.5) of $\underline{Y}^{(\infty)}$ to continue about $x = \infty$. If we set $\underline{W} = \mathrm{col}(W_j)$, $U = \mathrm{diag}(H_{1j})$ and $K_{jk} = \exp(2\pi i a_k(j-1)$, j, $k = 1,\ldots, m$, then it is found that (4.7.11) is equivalent to

$$\underline{W} = KUH^{-1}\underline{Y}^{(0)}. \tag{4.7.13}$$

To find the monodromy matrix at $x = 0$, V_0, we continue (4.7.13) round $x = 0$ to obtain, with the help of (4.7.4),

$$\underline{W}^c = KUH^{-1}C^2\underline{Y}^{(0)} \tag{4.7.14}$$

so that with

$$V_0 = KUH^{-1}C^2HU^{-1}K^{-1} \tag{4.7.15}$$

we have

$$\underline{W}^c = V_0 \underline{W}, \qquad x = 0. \tag{4.7.16}$$

Using (4.7.7), we have

$$V_0 = KTG^{-1}C^2GT^{-1}K^{-1} \tag{4.7.17}$$

where $T = \mathrm{diag}(G_{1j})$, so that $U = TA$.

In the same way, using

$$\underline{W} = KU\underline{Y}^{(\infty)}, \tag{4.7.18}$$

we find that V_∞, the monodromy matrix at $x = \infty$, is given by

$$V_\infty = KA^2K^{-1}. \tag{4.7.19}$$

The monodromy matrix at $x = 1$ may now be found, since the product of the three monodromy matrices must equal the unit matrix [12].

The Riemann module [12, 13] consists of all m-plets such as $\underline{W}$ for which each component is analytic in the cut x-plane, and which continue round the branch points with the same monodromy matrices. Any $(m + 1)$ such m-plets are related linearly with coefficients polynomial in x. It is shown in Appendix 1 that V_0 and V_∞ are unchanged if integers are added to $a_1, ..., a_m$, $c_2, ..., c_m$, so that all functions contiguous with (4.7.2) belong to the same module.

We define the dual module as the module based on the same singular points but with monodromy matrices $\tilde{V}$ related to the original V by

$$\tilde{V} = (V^T)^{-1}. \tag{4.7.20}$$

We aim to determine a set of functions in the module dual to the one above. We let $\bar{G}$ denote the matrix G when the signs of $a_1, ..., a_m$, $c_2, ..., c_m$ have been changed, and similarly for other matrices. Note that $\bar{A} = A^{-1}$, $\bar{C} = C^{-1}$.

Let us define matrix J as

$$J = TG^{-1}C^2GT^{-1}. \tag{4.7.21}$$

In Appendix 1 we show that a diagonal matrix D exists such that

$$DJD^{-1} = (\bar{T}\bar{G}^{-1}C^2\bar{G}\bar{T}^{-1})^T, \tag{4.7.22}$$

from which it follows that

$$\begin{aligned}\tilde{V}_0 = (V_0^T)^{-1} &= (K^T)^{-1}\,(J^T)^{-1}\,K^T \\ &= (K^T)^{-1}\,D\bar{T}\bar{G}^{-1}\bar{C}^2\bar{G}\bar{T}^{-1}D^{-1}K^T \\ &= S\bar{V}_0S^{-1},\end{aligned} \tag{4.7.23}$$

where

$$S = (K^T)^{-1} D\bar{K}^{-1}. \tag{4.7.24}$$

Similarly, we find that

$$\begin{aligned}\tilde{V}_\infty &= (V_\infty^T)^{-1} \\ &= S\bar{V}_\infty S^{-1}.\end{aligned} \tag{4.7.25}$$

Thus, since

$$\underline{\bar{W}}^c = \bar{V}\underline{\bar{W}} \tag{4.7.26}$$

we find that

$$\underline{\tilde{W}}^c = \tilde{V}\underline{\tilde{W}} \tag{4.7.27}$$

with

$$\underline{\tilde{W}} = S\underline{\bar{W}} \tag{4.7.28}$$

This gives an explicit expression $\underline{\tilde{W}}$ for elements of the dual module. We note that S is unchanged if we pass to a contiguous function so that any function contiguous to (4.7.2) may be used to generate an element in the dual module.

4.7.2. *Hermite–Padé Polynomials—Example.* We use the ideas of Chudnovsky [12] to construct H–P polynomials in the diagonal case with $m = 3$ and

$$\begin{aligned}F_1(x) &= {}_3F_2(a_1, a_2, a_3; c_2, c_3; x) \\ F_2(x) &= {}_3F_2(a_1, a_2 + 1, a_3; c_2 + 1, c_3; x) \\ F_3(x) &= {}_3F_2(a_1, a_2, a_3 + 2; c_2, c_3 + 2; x).\end{aligned} \tag{4.7.29}$$

We denote by $\underline{W}(j)$, $j = 1, 2, 3$, the elements in the module corresponding to $F_j(x)$ above. Additionally we define $\underline{W}(j)$, $j = 4, 5$, so that

$$\begin{aligned}W_1(4; x) = {}_3F_2(&a_1 + 2n + 2, a_2 + 2n + 2, a_3 + 2n + 2; \\ &c_2 + 3n + 3, c_3 + 3n + 3; x) \\ W_1(5; x) = {}_3F_2(&a_1 + 2n + 3, a_2 + 2n + 3, a_3 + 2n + 3; \\ &c_2 + 3n + 4, c_3 + 3n + 5; x).\end{aligned} \tag{4.7.30}$$

The general theory [12, 13] shows that rational functions $\pi_j(x)$, $j = 1, 2, 3$, exist such that, for given λ, μ,

$$\sum_{j=1}^{3} \underline{W}(j)\, \pi_j(x) = \lambda x^{3n+2} \underline{W}(4) + \mu x^{3n+3} \underline{W}(5). \tag{4.7.31}$$

Indeed, we may solve (4.7.31) for $\pi_j(x)$ to obtain

$$\pi_1(x) = \det |(\lambda_4 x^{3n+2} \underline{W}(4) + \lambda_5 x^{3n+3} \underline{W}(5)), \underline{W}(2), \underline{W}(3)| \\ /\det | \underline{W}(1), \underline{W}(2), \underline{W}(3)| \qquad (4.7.32)$$

and so on.

In $\det |\underline{W}(1), \underline{W}(2), \underline{W}(3)|$ we replace each $\underline{W}(j)$ by (4.7.13), and then (4.7.4), with the help of (A1.12), shows that, near $x = 0$,

$$\det |\underline{W}(1), \underline{W}(2), \underline{W}(3)| = x^{1-(c_2+1)+1-(c_3+2)} \times (\text{analytic in } x) \\ = x^{-c_2-c_3-1} \times (\text{analytic in } x). \qquad (4.7.33)$$

Similarly, near infinity,

$$\det |\underline{W}(1), \underline{W}(2), \underline{W}(3)| = x^{-a_1-a_2-a_3} \times (\text{analytic in } x^{-1}). \quad (4.7.34)$$

We now wish to find the behavior near $x = 1$. Pochhamer [39] showed that three independent solutions of (4.7.3) exist of the form

$$(x-1)^s \left(\sum_{k=0}^{\infty} b_k (x-1)^k \right), \qquad b_0 = 1 \qquad (4.7.35)$$

where $s = 0, 1, d = c_2 + c_3 - a_1 - a_2 - a_3$. We note that d is the same for all $\underline{W}(j), j = 1,..., 5$. A matrix P exists for which

$$P\underline{W}(1) = \begin{pmatrix} w_1 \\ (x-1)\, w_2 \\ (x-1)^d\, w_3 \end{pmatrix} \qquad (4.7.36)$$

where $w_j(x)$, $j = 1, 2, 3$ are functions analytic and non-zero at $x = 1$. However, because two exponents differ by an integer, a similar form for $\underline{W}(j), j = 2,..., 5$, using the same P, will not contain the factor $(x - 1)$ in the second row. We deduce that

$$\det |\underline{W}(1), \underline{W}(2), \underline{W}(3)| = (x-1)^d \times (\text{analytic at } x = 1). \quad (4.7.37)$$

It follows that

$$\det |\underline{W}(1), \underline{W}(2), \underline{W}(3)| = \alpha x^{-c_2-c_3-1} (x-1)^d\, (x-\beta). \qquad (4.7.38)$$

In the same way we find that the determinant in the numerator of $\pi_1(x)$ in (4.7.32) has the form $x^{-c_2-c_3-1}(x-1)^d \times$ (polynomial of degree $(n+1)$), and similarly for $\pi_2(x)$, $\pi_3(x)$.

If we choose λ_4/λ_5 so that the numerator of $\pi_1(x)$ vanishes at $x = \beta$, then so will the numerators of $\pi_2(x)$, $\pi_3(x)$, since for that value of x the columns $\underline{W}^{(1)}$, $\underline{W}^{(2)}$, $\underline{W}^{(3)}$ are linearly dependent. With this choice of λ_4/λ_5 the functions $\pi_j(x)$, $j=1,2,3$ are polynomials of degree n which are the H–P polynomials of type I for the functions $F_j(x)$, $j=1,2,3$, on account of the fact that the right-hand side of (4.7.31) is $O(x^{3n+2})$ as $x \to 0$ on the first sheet.

It is seen that the term containing $\underline{W}(5)$ in (4.7.31) has to be added to Chudnovsky's form of the remainder [12] in order to make it correct.

The construction of the type II polynomials is based on the fact that, if $\underline{X}$, $\underline{\tilde{Z}}$ are any elements from the module and its dual, respectively, then $\underline{X}^T\underline{\tilde{Z}}$ is a rational function of x with poles possible only at $x=0$, 1, ∞ (see Appendix 1). We introduce $\underline{W}(j)$, $j=6$, 7, 8, so that

$$\begin{aligned} W_1(6;x) &= {}_3F_2(a_1+2n, a_2+2n, a_3+2n; c_2+3n, c_3+3n; x) \\ W_1(7;x) &= {}_3F_2(a_1+2n, a_2+2n, a_3+2n; c_2+3n-1, c_3+3n+1; x) \\ W_1(8;x) &= {}_3F_2(a_1+2n-1, a_2+2n, a_3+2n; c_2+3n-1, c_3+3n; x) \end{aligned} \tag{4.7.39}$$

and define

$$\xi_j(x) = M_{11}(j)^{-1}\,\underline{W}^T(j)\left(\sum_{k=6}^{8} \lambda_k \underline{\tilde{W}}(k)\right), \qquad j=1,2,3 \tag{4.7.40}$$

with $M_{11}(j)$ given by (A1.13). The constants λ_k, $k=6,7,8$ are still to be determined.

The argument of Appendix 1 shows that $\xi_j(x)$, $j=1,2,3$, are polynomials of degree $2n$. In view of (A1.15), the may be written in the form (for $\bar{B}_l$ see (A1.17))

$$\xi_j(x) = \sum_{k=6}^{8} \lambda_k \sum_{l=1}^{3} M_{1l}(j)\,M_{11}(j)^{-1}\,y_l^{(0)}(j;x)\,\bar{y}_l^{(0)}(k;x)\,\bar{B}_l\bar{M}_{ll}(k). \tag{4.7.41}$$

Thus

$$\begin{aligned} &F_i(x)\,\xi_j(x) - F_j(x)\,\xi_i(x) \\ &\quad = \sum_{k=6}^{8} \lambda_k \sum_{l=2}^{3} [M_{1l}(j)\,M_{11}(j)^{-1}\,y_1^{(0)}(i;x)\,y_l^{(0)}(j;x) \\ &\qquad - M_{1l}(i)\,M_{11}(i)^{-1}\,y_1^{(0)}(j;x)\,y_l^{(0)}(i;x)]\,\bar{y}_l^{(0)}(k;x)\,\bar{B}_l\bar{M}_{ll}(k), \\ &\qquad\qquad i,j=1,2,3. \end{aligned} \tag{4.7.42}$$

Using the expansion (4.7.4) we see that the right-hand side of (4.7.42) is in every case $O(x^{3n})$ for $x \approx 0$. If λ_k, $k = 6, 7, 8$ are chosen so that the coefficient of x^{3n} in the expansion of (4.7.42) for $i, j = 1, 2;\ 1, 3$, is zero, then $\xi_j(x)$ will be the type II H–P polynomials of degree $2n$ for the functions $F_j(x)$; $j = 1, 2, 3$. This leads to

$$\begin{aligned} \lambda_6 &= -\lambda_8 \bar{M}_{33}(8)/\bar{M}_{33}(6) \\ \lambda_7 &= -\lambda_8 \bar{M}_{22}(8)/\bar{M}_{22}(7), \end{aligned} \tag{4.7.43}$$

or from (A1.13)

$$\begin{aligned} \lambda_6 &= \frac{(c_2 + 3n)(c_3 - a_1 + n + 1)}{(a_1 + 2n)(c_2 - c_3 - 1)} \lambda_8 \\ \lambda_7 &= -\frac{(c_3 + 3n + 1)(c_2 - a_1 + n)}{(a_1 + 2n)(c_2 - c_3 - 1)} \lambda_8. \end{aligned} \tag{4.7.44}$$

A consequence of the above remarks is that (4.7.41) may be replaced by ($\bar{M}_{11}$ is invariant)

$$\xi_j(x) = \bar{B}_1 \bar{M}_{11} \sum_{k=6}^{8} y_1^{(0)}(j; x)\, \bar{y}_1^{(0)}(k; x)\,|^{2n} \tag{4.7.45}$$

where it is meant that we expand in powers of x and cut off the expansion after the term containing x^{2n}.

4.7.3. *Asymptotic Behavior.* To obtain the asymptotic form of the type I polynomials of Section 4.7.2 we use, after Chudnovsky [12], the Pochhammer integral representation [39]

$$\begin{aligned} &{}_3F_2(a_1 + 2n, a_2 + 2n, a_3 + 2n;\ c_2 + 3n, c_3 + 3n;\ x) \\ &\quad = \left(\prod_{j=2}^{3} \frac{\Gamma(c_j + 3n)}{\Gamma(a_j + 2n)\,\Gamma(c_j - a_j + n)} \right) \\ &\quad\quad \times \int_0^1 dt_2 \int_0^1 dt_3 \left(\prod_{j=2}^{3} (1 - t_j)^{c_j - a_j + n - 1}\, t_j^{a_j + 2n - 1} \right) (1 - t_2 t_3 x)^{-a_1 - 2n} \end{aligned} \tag{4.7.46}$$

The integral we write in the form

$$I = \int_0^1 dt_2 \int_0^1 dt_3 \left(\prod_{j=2}^{3} (1 - t_j)^{c_j - a_j - 1}\, t_j^{a_j - 1} \right) (1 - t_2 t_3 x)^{-a_1} J^n \tag{4.7.47}$$

where

$$J = \left(\prod_{j=2}^{3} (1 - t_j)\, t_j^2 \right) (1 - t_2 t_3 x)^{-2} \tag{4.7.48}$$

and apply the saddle-point method [8]. This requires solving the equations

$$\begin{aligned} \frac{\partial \log J}{\partial t_2} &= \frac{-1}{1-t_2} + \frac{2}{t_2} + \frac{2t_3 x}{1-t_2 t_3 x} = 0 \\ \frac{\partial \log J}{\partial t_3} &= \frac{-1}{1-t_3} + \frac{2}{t_3} + \frac{2t_2 x}{1-t_2 t_3 x} = 0 \end{aligned} \tag{4.7.49}$$

giving $t_2 = t_3 = y$, where y satisfies

$$y^3 x - 3y + 2 = 0. \tag{4.7.50}$$

For large n the approximate value of (4.7.47) is obtained by evaluating the integrand of (4.7.47) at these values of t_2, t_3 and multiplying by

$$\text{const.}\det \begin{bmatrix} \dfrac{\partial^2 \log J}{\partial^2 t_2} & \dfrac{\partial^2 \log J}{\partial t_2\, \partial t_3} \\ \dfrac{\partial^2 \log J}{\partial t_3\, \partial t_3} & \dfrac{\partial^2 \log J}{\partial^2 t_3} \end{bmatrix}^{-1/2} \tag{4.7.51}$$

where the derivatives are evaluated at $y_2 = y_3 = y$. We find

$$I \underset{n\to\infty}{\sim} \text{const.}\, y^{6n} \left(\prod_{j=2}^{3} (1-y)^{c_j - a_j} y^{a_j} \right) (1-y)^{-1/2 - a_1} y^{a_1 - 1} \tag{4.7.52}$$

where the constant is the product of the constant in front of the integral in (4.7.46) and a factor independent of $\{a_j\}$, $\{c_j\}$.

Before proceeding, let us investigate (4.7.50), which describes the Riemann surface $\mathscr{R}$ of three sheets

$$y^3 - 3yz + 2z = 0. \tag{4.7.53}$$

This surface has genus 0 with square root branch points at $z = 1, \infty$ and a cube root branch point at $z = 0$. The function $\phi(z)$ constructed according to the procedure of Section 3.1 is given by

$$\exp(\phi(z)) = 27^{-1} x^2 y^6, \tag{4.7.54}$$

and so the sheets are chosen according to value of $|y|$. With the notation of Section 3.1 we find that the boundaries between adjacent sheets all correspond to real z. We have

$$\begin{aligned} S' &= \{\text{boundary between sheets } 1, 2\} = \{z\colon 0 \leqslant z \leqslant 1\} \\ S &= \{\text{boundary between sheets } 2, 3\} = \{z\colon -\infty < z \leqslant 0\}. \end{aligned} \tag{4.7.55}$$

Now the analysis of Section 4.7.2 shows that for the diagonal polynomials $\{p_j(z)\}$ (defined in (3.2.2)) corresponding to $F_j(x), j = 1, 2, 3$, of (4.7.29) the exact remainder function $\bar{R}(z)$ is given by

$$\sum_{j=1}^{3} F_j(x)\, p_j(z) = \bar{R}(z) \tag{4.7.56}$$

where

$$\bar{R}(z) = x^{2n+2}(\lambda_4 W_1(4; x) + \lambda_5 x W_1(5; x)). \tag{4.7.57}$$

In order to work out the asymptotic form of $\bar{R}(z)$ for $z \approx \infty$, corresponding to $F_j(x) \approx 1, j = 1, 2, 3$, we may use (4.7.52) in (4.7.30) and analysis shows that for y in (4.7.52) we must choose y_1, the value on sheet 1, which is the solution of (4.7.50) with smallest modulus. The continuation of (4.7.52) presumably gives an approximation to the analytic continuation of $\bar{R}(z)$ so long as we continue along a path for which $\operatorname{Re}\phi(z)$ is non-decreasing. The situation is similar to Section 4.6 but now we have a different surface.

Now it is clear that, if $z \notin S$, $\bar{R}(z^{(3)}) \gg \bar{R}(z^{(2)}) \gg \bar{R}(z^{(1)})$, where the various $\bar{R}(z)$ are evaluated by continuing along a path as above. Writing (4.7.56) for $z^{(k)}$, $k = 1, 2, 3$, and solving for $p_j(z)$, we find, with the notation of Section 3.2,

$$p_j(z) \underset{n\to\infty}{\sim} \chi_j(z) = D^{-1}(z)\, A_j(z)\, R(z), \qquad z \in \mathscr{R}_3, \tag{4.7.58}$$

where, using (4.7.56) and (4.7.52),

$$R(z) = x^{2n+2} y^{6n + \sum_{j=1}^{3} a_j + 5} (1-y)^{d-1/2}\, (y^3 - y_3^3(\beta)), \tag{4.7.59}$$

with

$$d = \sum_{j=2}^{3} c_j - \sum_{j=1}^{3} a_j \tag{4.7.60}$$

and $y(\beta)$ is the solution of (4.7.50) with $x = \beta$.

The function $R(z)$ is an approximation to $\bar{R}(z)$, neither of which are single valued, $z \in \mathscr{R}$. In $\mathscr{R}_0$ the functions $\{f_j(z)\}$ and $R(z)$, $\bar{R}(z)$ are meromorphic except for a branch point at $z = 1$, the branch point of $\mathscr{R}$ where sheets 1, 2 meet, which corresponds to $y = 1$. We insert a cut σ as described in Section 3.3 from this point to s. We choose σ to lie on sheet 2 and run from 1 to $+\infty$ along the real axis in the z-plane, which is equivalent to the same set in the y-plane.

To make the above functions single-valued on $\mathscr{R}$ in order to facilitate the analysis, we insert a cut ρ on sheet 3 along the real axis in the z-plane from 0 to $+\infty$, which corresponds to the negative real axis in the y-plane. With these cuts, we expect $\bar{R}(z) \sim R(z)$ will hold throughout $\mathscr{R}$ except near branch points.

Now for the choice (4.7.29), (3.3.1) holds with $l = 2$ at the point v given by $y = 1$. We shall show that the conjecture of Section 3.3. holds with $\mathscr{R}$ of (4.7.53) and σ as described above. We must demonstrate that $\chi_j(z)$ given by (4.7.58) is analytic, $z \in \mathscr{R}_3$, which means that it must be shown to be continuous across the cut ρ. Let z be real, $0 < z < 1$, and let $z^{(k)}+$, $z^{(k)}-$ denote points on sheet k just above, below the point z (i.e., with small positive, negative imaginary parts). Then we have

$$\begin{aligned} f_j(z^{(1)}+) &= f_j(z^{(2)}-) \\ f_j(z^{(2)}+) &= f_j(z^{(1)}-) \end{aligned} \qquad j = 1, 2, 3 \tag{4.7.61}$$

which means that $A_j(z^{(3)}+) = A_j(z^{(3)}-)$. Also, if we continue clockwise once round $z = 0$, we find

$$\begin{aligned} &f_j(z^{(3)}-) \to f_j(z^{(2)}+) \\ &f_j(z^{(1)}-) \to f_j(z^{(1)}+), \qquad j = 1, 2, 3 \\ &f_j(z^{(2)}-) \to f_j(z^{(3)}+) \end{aligned} \tag{4.7.62}$$

and so

$$D(z^{(3)}-) \to -D(z^{(3)}+). \tag{4.7.63}$$

The continuation of $D(z)$ may be carried out with the help of (4.7.38) and of $R(z)$ (round $y = 0$) from (4.7.59), and it is easily seen that $\chi_j(z)$ has no discontinuity across that part of ρ, $0 < z < 1$.

Now $R(z)$ is not singular at $z = 1$, $z \in \mathscr{R}_3$, and (4.7.36) shows that the ratio $A_j(z)/D(z)$ is analytic there, so that $\chi_j(z)$ is continuous across the whole of ρ. It follows from (4.7.58) that the equations (3.2.4), (3.2.5) of the conjecture are satisfied.

From (4.7.59) we see that (3.3.2) holds with $\lambda(z)$ that is derived from (3.31), thus verifying this part of the conjecture.

We do not know of an integral representation that may be used to analyze the asymptotics of the generalized hypergeometric functions needed for the type II polynomials. However these functions satisfy third order differential equations with polynomial coefficients and we expect that the same formal expansion methods which give the correct result for (4.7.46) will still work. That is we predict that (4.7.52) remains valid if we change the sign of n. With this assumption we can show that the conjecture of Section 3.3 is valid for the type II polynomials of Section 4.7.2.

5. Heuristic and Numerical Results

In Sections 5.1, 5.3, heuristic treatments of asymptotics for some interesting examples are given (the work of Section 5.1 is perhaps not far from being made rigorous). The results are in agreement with the conjecture and are supported by numerical examples. Section 5.2 reviews an example where the predictions of the conjecture are compared with numerical work.

5.1. *Generalized Jacobi Polynomials*

In this section we study the asymptotic behavior of polynomials $p_1(z)$, $p_2(z)$ of degree n corresponding to the case $m = 2, f_1(z) = 1$ and

$$f_2(z) = \prod_{j=1}^{3} (z - b_j)^{\nu_j} \tag{5.1.1}$$

where the non-integers ν_j satisfy

$$\sum_{j=1}^{3} \nu_j = 0. \tag{5.1.2}$$

Thus we have

$$p_1(z) + f_2(z)\, p_2(z) = O(z^{-(n+1)}) \qquad \text{as} \quad z \to \infty^{(1)} \tag{5.1.3}$$

and $p_2(z)$ is the orthogonal polynomial for weight $f_2(z)$ in the sense (1.2.6).

The functions $1, f_2(z)$ form a basis for a Riemann module (Chudnovsky [12, 13]), and it is our hope that, if successful, the method outlined here for determining the asymptotic behavior of $p_1(z)$, $p_2(z)$ may be extended to the important general case of Hermite–Padé approximation to functions taken from a Riemann module of dimension m.

The case at hand was first studied, in slightly more generality, by Laguerre [25]. He showed that $p_2(z)$ satisfies a differential equation with polynomial coefficients and obtained non-linear recurrence relations relating the coefficients in this equation and the coefficients in the three-term relation connecting polynomials with adjacent degrees. He was unable to solve the recurrence relations or to obtain the asymptotic behavior of the coefficients. A conjecture about this asymptotic behavior was given by Gammel and Nuttall [16]. This conjecture, supported by numerical evidence, proposed that the coefficients asymptotically were related to certain elliptic functions of n. Approximate solution of the differential equation led to a result for $p_2(z)$ consistent with the conjecture of Section 3.

We now propose an alternative method of proof, which although at present containing some gaps, appears to be more promising than the approach based on the Laguerre recurrence relations. It is based on the rigorous treatment of error bounds in the Liouville–Green, (LG, sometimes called WKB) method described by Olver [38].

First we obtain the form of the differential equation satisfied by $p_2(z)$ and $R(z)$ defined by

$$f_2^{-1}(z)\,p_1(z)+p_2(z)=R(z) \tag{5.1.4}$$

using a technique described by Chudnovsky [12]. We suppose $f_2(z)$ is made single valued by inserting cuts joining the branch points b_1, b_2, b_3 and the let $R(z^{(1)})$ denote the value of $R(z)$ on the first sheet (the one we use in (5.1.3)). Let $R(z^{(2)})$ be a continuation of $R(z^{(1)})$ through a cut, so that $R(z^{(2)})$ is single valued in the cut plane.

Then, with $'$ meaning differentiation, we have

$$\det\begin{vmatrix} R'(z^{(1)}), R(z^{(1)}) \\ R'(z^{(2)}), R(z^{(2)}) \end{vmatrix} \begin{pmatrix} R''(z^{(1)}) \\ R''(z^{(2)}) \end{pmatrix} - \det\begin{vmatrix} R''(z^{(1)}), R(^{(1)}) \\ R''(z^{(2)}), R(z^{(2)}) \end{vmatrix} \begin{pmatrix} R'(z^{(1)}) \\ R'(z^{(2)}) \end{pmatrix}$$
$$+\det\begin{vmatrix} R''(z^{(1)}), R'(z^{(1)}) \\ R''(z^{(2)}), R'(z^{(2)}) \end{vmatrix} \begin{pmatrix} R(z^{(1)}) \\ R(z^{(2)}) \end{pmatrix} = 0. \tag{5.1.5}$$

The determinants are analytic in the cut plane except at the branch points and possibly at ∞. From (5.1.4) and the behavior $R(z^{(1)}) \sim z^{-n-1}$, $R(z^{(2)}) \sim z^n$ near ∞ we deduce that

$$X(z)\,f_2(z)\det\begin{vmatrix} R'(z^{(1)}), R(z^{(1)}) \\ R'(z^{(2)}), R(z^{(2)}) \end{vmatrix} = \pi_1(z) \tag{5.1.6}$$

$$X^2(z)\,f_2(z)\det\begin{vmatrix} R''(z^{(1)}), R(z^{(1)}) \\ R''(z^{(2)}), R(z^{(2)}) \end{vmatrix} = \pi_3(z) \tag{5.1.7}$$

$$X^2(z)\,f_2(z)\det\begin{vmatrix} R''(z^{(1)}), R'(z^{(1)}) \\ R''(z^{(2)}), R'(z^{(2)}) \end{vmatrix} = \pi_2(z) \tag{5.1.8}$$

where $\pi_j(z)$, $j=1,2,3$, is a polynomials of degree j, and $X(z)=\prod_{j=1}^{3}(z-b_j)$. Thus $R(z)$ and also, it may be deduced, $p_2(z)$, satisfy an equation of the form

$$X(z)(z-z_n)\,R''(z)+\pi_3(z)\,R'(z)+\pi_2(z)\,R(z)=0. \tag{5.1.9}$$

The form of the solutions of (5.1.9) near the branch points and ∞ places restrictions on $\pi_2(z)$, $\pi_3(z)$ and we find that

$$\pi_3(z)=(z-z_n)\,Z(z)-X(z) \tag{5.1.10}$$

$$\pi_2(z)=-n(n+1)(z-v_n)(z-a_n) \tag{5.1.11}$$

where the degree 2 polynomial $Z(z)$ is given by

$$Z(z)\, f_2(z) = (X(z)\, f_2(z))'. \tag{5.1.12}$$

The differential equation (5.1.9) has a singular point at z_n, but, since no solution of (5.1.9) is singular there, this must be an apparent singularity [21] which leads to the relation

$$\begin{aligned} n(n+1)(v_n - z_n)^2\,(a_n - z_n)^2 + (v_n - z_n)(a_n - z_n)(X'(z_n) - Z(z_n)) \\ + (v_n + a_n - 2z_n)\, X(z_n) = 0 \end{aligned} \tag{5.1.13}$$

If we take $|v_n - z_n| < |a_n - z_n|$ then it can be shown from (5.1.13) that

$$|v_n - z_n| < \text{const.}(|z_n| + 1)\, n^{-2/3} \tag{5.1.14}$$

To apply the LG method we remove the R' term from (5.1.9) with the substitution

$$R(z) = (z - z_n)^{1/2}\,(X(z)\, f_2(z))^{-1/2}\, u(z) \tag{5.1.15}$$

so that

$$u''(z) = \lambda(z)\, u(z) \tag{5.1.16}$$

where

$$\lambda(z) = \frac{n(n+1)(z - a_n)(z - v_n)}{X(z)(z - z_n)} - \left[\frac{\pi_3(z)}{2(z - z_n)\, X(z)}\right]' - \left[\frac{\pi_3(z)}{2(z - z_n)\, X(z)}\right]^2. \tag{5.1.17}$$

At this point we interrupt the development to note an interesting observation made by G. Chudnovsky. Suppose we choose $b_1 = 0$, $b_2 = 1$ and $b_3 = b$. Then the differential equation relating z_n to b is the Painlevé equation

$$\begin{aligned} \frac{d^2 z}{db^2} = \frac{1}{2}\left\{\frac{1}{z} + \frac{1}{z-1} + \frac{1}{z-b}\right\}\left(\frac{dz}{db}\right)^2 - \left\{\frac{1}{b} + \frac{1}{b-1} + \frac{1}{z-b}\right\}\frac{dz}{db} \\ + \frac{z(z-1)(z-b)}{b^2(b-1)^2}\left\{\bar{\alpha} + \frac{\bar{\beta} b}{z^2} + \frac{\bar{\gamma}(b-1)}{(z-1)^2} + \frac{\bar{\delta} b(b-1)}{(z-b)^2}\right\} \end{aligned} \tag{5.1.18}$$

$$\bar{\alpha} = -2n(n+1) + \text{const.}, \qquad \bar{\beta}, \bar{\gamma}, \bar{\delta} = \text{const.} \tag{5.1.19}$$

where the constants are independent of n. This follows immediately from the work of Fuchs [15, 11].

Now we quote the results of Olver's work [38] on the LG method as applied to this case. We set

$$f(z) = \frac{(n+1/2)^2\,(z-a_n)(z-v_n)}{X(z)(z-z_n)}. \tag{5.1.20}$$

and

$$g(z) = \lambda(z) - f(z), \tag{5.1.21}$$

and define

$$H(z) = \int_{\infty^{(1)}}^{z} dt \left[f^{-1/4}(t) \frac{d^2}{dt^2} (f^{-1/4}(t)) - g(t) f^{-1/2}(t) \right] \tag{5.1.22}$$

Then the solution of (5.1.16) with the behavior z^{-n} at $\infty^{(1)}$ may be written

$$u(z) = f^{-1/4}(z) \exp[-(n+1/2)\,\psi(z)](1+\varepsilon(z)) \tag{5.1.23}$$

where

$$\psi(z) = \int_{z_\infty}^{z} dt \left[\frac{(t-a_n)(t-v_n)}{X(t)(t-z_n)} \right]^{1/2}. \tag{5.1.24}$$

Provided that path followed from $\infty^{(1)}$ to z in the integral (5.1.22) is a progressive path, it may be shown [38] that

$$|\varepsilon(z)| \leqslant \exp\{V_z(H)\} - 1 \tag{5.1.25}$$

where the variation $V_z(H)$ is

$$V_z(H) = \int_{\infty^{(1)}}^{z} |dt| \left| \frac{dH(t)}{dt} \right| \tag{5.1.26}$$

with the integral taken along the same path as above. A progressive path is an adequately smooth path along which $\operatorname{Re}\psi(z)$ is non-increasing as z moves away from $\infty^{(1)}$.

Now to proceed we suppose that there is no subsequence of the integers n for which $a_n \to b_j$, $j = 1, 2$ or 3, or $a_n \to \infty$. We postulate the existence for each n of a point z_0 and progressive paths Γ_k, $k = 1, 2, 3$, from $\infty^{(1)}$ to z_0, giving rise to values $\psi(z_0^{(k)})$. The paths are such that $\Gamma_2\Gamma_1^{-1}$ encircles b_2 once in a counter-clockwise direction and $\Gamma_3\Gamma_1^{-1}$ encircles b_3 once in a clockwise direction, but neither loop includes b_1 or a_n, and they do not pass close to z_n, but each path passes through a point z_∞ near $\infty^{(1)}$. There must exist constants $\delta_1, \delta_2 > 0$ such that

$$V_{z_0}(H) \leqslant \delta_2 n^{-1} \qquad \text{for each path } \Gamma_j \tag{5.1.27}$$

and

$$\operatorname{Re} \psi(z_0^{(1)}) - \operatorname{Re} \psi(z_0^{(2)}) \geqslant \delta_1. \tag{5.1.28}$$

Similar paths must exist for the pairs b_1, b_2 and b_3, b_1, these points being relabelled if necessary in order to make possible the construction of the paths.

To use this conjecture we write from (5.1.4)

$$f_2^{-1}(z_0^{(j)})\, p_1(z_0) + p_2(z_0) = R(z_0^{(j)}), \qquad j = 1, 2, 3 \tag{5.1.29}$$

and eliminate $p_1(z_0)$, $p_2(z_0)$ to obtain

$$\begin{aligned} R(z_0^{(3)})/R(z_0^{(2)}) = \{[f_2^{-1}(z_0^{(3)}) - f_2^{-1}(z_0^{(1)})] + R(z_0^{(1)})/R(z_0^{(2)})\} \\ /(f_2^{-1}(z_0^{(2)}) - f_2^{-1}(z_0^{(1)})). \end{aligned} \tag{5.1.30}$$

On substituting from (5.1.1), (5.1.15) and (5.1.23) into (5.1.30) we obtain, assuming that $\Gamma_2 \Gamma_1^{-1}$ does not contain z_n,

$$\exp\{(n + 1/2)(\psi(z_0^{(2)}) - \psi(z_0^{(3)}))\} = (\sin \pi v_3)/(\sin \pi v_2) + O(n^{-1}), \tag{5.1.31}$$

since, from (5.1.28), $R(z_0^{(1)})/R(z_0^{(2)}) \to 0$ as $n \to \infty$ much faster than n^{-1}. Two similar equations may be obtained by considering the other pairs of points. In view of (5.1.14) we may write, using part of Γ_k to evaluate the integral,

$$\psi(z_0^{(k)}) = \int_{z_\infty}^{z_0} dt \left[\frac{(t - a_n)}{X(t)}\right]^{1/2} + O(n^{-2/2}), \qquad k = 2, 3 \tag{5.1.32}$$

so that (5.1.31) gives, on distorting the contours,

$$\operatorname{Re} \int_{b_2}^{b_3} dt \left[\frac{(t - a_n)}{X(t)}\right]^{1/2} = O(n^{-2/3}) \tag{5.1.33}$$

and similarly for $b_1 b_2, b_3 b_1$.

It is known [18, 36] that the equations obtained by replacing the right-hand side of (5.1.33) by zero have solutions $a_n = b_1$, b_2, b_3 or $a_n = a$ (which might be called the center of capacity) and no other. It follows that a_n approaches one of b_1, b_2, b_3, a possibility we have ruled out for now, or that, from the implicit function theorem,

$$a_n = a + O(n^{-2/3}). \tag{5.1.34}$$

(We must assume that the points $b_1 b_2 b_3$ are not collinear, so that a does not coincide with any of them.)

Now to continue we assume that z_n is not near $b_j, j=1, 2, 3$, or a or ∞. Using (5.1.34) in (5.1.13) gives

$$(z_n - v_n)^2 = n^{-2} \frac{X(z_n)}{z_n - a} + O(n^{-3}) \tag{5.1.35}$$

which incidentally makes it possible to improve the error in (5.1.14) to $O(n^{-1})$. For use in (5.1.31) we expand the integrand of (5.1.24) as

$$\left[\frac{(t-a_n)(t-v_n)}{X(t)(t-z_n)}\right]^{1/2} = \frac{(t-a)^{1/2}}{X(t)^{1/2}} \left[1 + \frac{1}{2}\,\frac{a-a_n}{t-a} + \frac{1}{2}\,\frac{z_n - v_n}{t - z_n} + O(n^{-2})\right] \tag{5.1.36}$$

so that (5.1.31) gives

$$\begin{aligned}
&(n+\tfrac{1}{2}) \int dt(t-a)^{1/2} X^{-1/2}(t) + \tfrac{1}{2}(n+\tfrac{1}{2})(a-a_n) \int dt[X(t)(t-a)]^{-1/2} \\
&\quad + \frac{1}{2}\left[\frac{X(z_n)}{z_n - a}\right]^{1/2} \int dt(t-a)^{1/2} X^{-1/2}(t)(t-z_n)^{-1} \\
&\quad - \log(\sin \pi v_3/\sin \pi v_2) + 2\pi i m_1 = O(n^{-1})
\end{aligned} \tag{5.1.37}$$

where m_1 is some integer. The contour in (5.1.37) is the loop $\Gamma_2\Gamma_3^{-1}$. There are similar equations for the other pairs b_1b_2, b_3b_1. From two of the equations we may eliminate a_n and obtain an equation for z_n, which, after some manipulation, may be put in the form

$$\begin{aligned}
\int_{\infty^{(1)}}^{z_n} dw_1 &= -(i\pi)^{-1} \int_S dz\, y_+^{-1}(z^{(1)}) \log \sigma(z) \\
&\quad - 2n \int_{\infty^{(1)}}^{b_1} dw_1 + \sum_{j=1}^{2} \eta_j \Omega_{1j} + O(n^{-1}).
\end{aligned} \tag{5.1.38}$$

We have used the notation of (4.3.25), with the surface $\mathscr{R}$ corresponding to $y^2(z) = X(z)(z-a)$. The set S is

$$S = \{z \in \mathbb{C} : \operatorname{Re} \phi(z) = 0\} \tag{5.1.39}$$

where

$$\phi(z) = \int_{b_1}^{z} dt(t-a)^{1/2} X^{-1/2}(t) \tag{5.1.40}$$

so that S consists of arcs running from a to each point b_j, $j=1, 2, 3$. The function $\sigma(z)$ is defined by

$$\sigma(z) = y_+(z^{(1)})(f_{2+}(z^{(1)}) - f_{2-}(z^{(1)})) \tag{5.1.41}$$

as in (4.3.27)

If we set $O(n^{-1})=0$ in (5.1.38), then there is a unique solution $\alpha_n \in \mathscr{R}$, so that if z_n is not near a or b_1, b_2, b_3 then z_n is near α_n. The situation is just as proposed by the conjecture. The approximation to $R(z)$ has the form

$$\begin{aligned} R_0(z) = {} & (z-\alpha_n)^{1/2} (X(z) f_2(z))^{-1/2} (z-a)^{-1/4} X^{1/4}(z) \\ & \exp \left\{ (n+\tfrac{1}{2}) \int_{z_\infty}^{z} dt (t-a)^{1/2} X^{-1/2}(t) [1 + A_n (t-a)^{-1}] \right. \\ & \left. + \tfrac{1}{2} X^{1/2}(\alpha_n)(\alpha_n - a)^{-1/2} \int_{z_\infty}^{z} dt (t-a)^{1/2} X^{-1/2}(t)(t-\alpha_n)^{-1} \right\}. \end{aligned} \tag{5.1.42}$$

The approximations to $p_j(z)$, namely $\chi_j(z)$, $j=1, 2$, analytic on sheet 2, are given by solving an approximation to (5.1.4) and its continuation,

$$f_2^{-1}(z^{(1)}) \chi_1(z) + \chi_2(z) = 0 \tag{5.1.43}$$

$$f_2^{-1}(z^{(2)}) \chi_1(z) + \chi_2(z) = R_0(z^{(2)}). \tag{5.1.44}$$

No matter which path is used to continue $R_0(z)$ across S onto sheet 2, it is ensured that the same $\chi_j(z)$ will be obtained.

To complete the discussion it is necessary to rule out the special cases not treated so far, but this remains to be done.

5.2. *Hypergeometric Functions—A Generalization of Section* 4.6

We extend the example of Section 4.6 and study the case

$$\begin{aligned} F_1(x) &= 1 \\ F_j(x) &= {}_2F_1(1, \omega_j; c_j; x), \qquad j = 2, \ldots, m \end{aligned} \tag{5.2.1}$$

where no two parameters in (5.2.1) differ by an integer. It is possible to show that these functions belong to a Riemann module, which leads to the hope of an eventual rigorous treatment (see Section 6.3), but here we mention the results of Baumel and Nuttall [4] for $m=3$. They compared the predictions of the conjecture for type I polynomials with the results of computer calculations.

We use the same surface $\mathscr{R}$ as in Section 4.6 given by (4.6.7). The homogeneous Hilbert problem (3.2.11) derived from the conjecture may be solved by applying the methods of Muskhelishvili [29] in the y-plane, which is a representation of $\mathscr{R}$. It was shown [4] that

$$\begin{aligned}\chi_j(z) = {} & (-2\pi i)^{-1} \exp(-i\pi\omega_j)(\Gamma(\omega_j)\,\Gamma(c_j-\omega_j)/\Gamma(c_j)) \\ & \times \lambda_j z^n (1-y_3)^{3n+2}\,(-y_3)^{\beta+3\omega_j} \\ & \times (1-\bar{\omega}y_3)^{\gamma}(1-\omega y_3)^{\delta}\,(1-z)^{1-c_j}, \qquad j=2,3 \end{aligned} \tag{5.2.2}$$

$$R(z) = z^n(1-y)^{3n+2}\,(-y)^{\beta}\,(1-\bar{\omega}y)^{\gamma}(1-\omega y)^{\delta} \tag{5.2.3}$$

where

$$\omega = \exp(2\pi i/3), \qquad \bar{\omega} = \exp(-2\pi i/3) \tag{5.2.4}$$

and

$$\lambda_3 = -\lambda_2 = \lambda \tag{5.2.5}$$

with

$$\beta = c_2 + c_3 - \omega_2 - \omega_3 - 3 \tag{5.2.6}$$

$$\gamma = 1 - \tfrac{1}{2}(c_2+c_3) + \mu \tag{5.2.7}$$

$$\delta = 1 - \tfrac{1}{2}(c_2+c_3) - \mu \tag{5.2.8}$$

$$\mu = (2\pi i)^{-1} \log\rho \tag{5.2.9}$$

$$\rho = \sin(\pi(\omega_3-\omega_2))/\sin(\pi(\omega_3-\omega_2+c_2-c_3)) \tag{5.2.10}$$

give a solution of the boundary value problem of the conjecture (3.2.4), (3.2.5) for the diagonal case.

The general solution is obtained by multiplying by an arbitrary rational function of y. Because points at ∞ lie on s and because these points and others on s are singularities of $f_j(z)$, $j=2,3$, we are unable to predict the form of this rational function. However, the calculations suggest that, in some cases, the form written is probably correct. In others, it seems to be necessary to multiply by a linear function of y. Some guesses about the form of this linear function are given in [4], but a complete understanding is still lacking.

5.3. *Saddle-Point Method*

The general structure of polynomial asymptotics presented in this paper was first suggested, as mentioned in [30] in the case $m=2$ by a heuristic treatment of the multiple integral representation of orthogonal polynomials, (2.2.10) of [45], which is a special case of (2.4.2), (2.4.5). In [3] we extended the idea to certain examples of type I polynomials in the case

$m = 3$. Here we review the previous work and consider how the results relate to the conjecture of Section 3.

We illustrate the idea by analyzing the case $m = 3$, although the method can be generalized to higher m with little difficulty. We suppose that $f_1(z) = 1$ and that $f_2(z)$ is analytic in the z-plane cut between b_1, b_2 and similarly for $f_3(z)$ with a cut joining b_3, b_4. We assume that these two cuts, locations determined later, do not intersect. If attention is restricted to the diagonal case, where $p_j(z)$, $j = 1, 2, 3$, are each of degree n, then $\{p_j(z)\}$ satisfying (3.2.3) are given, for $j = 2, 3$ by (2.4.2). If the contour of integration is contracted until it closely surrounds arcs Γ_2, Γ_3 joining b_1, b_2 and b_3, b_4, respectively, it is seen that the path of integration for variables $\{z_k^{(2)}\}$ may be taken to be Γ_2 with $\omega_2(z)$ replaced by the discontinuity $f_{2+}(z) - f_{2-}(z)$, and likewise for $j = 3$. We are assuming that the functions $\{f_j(z)\}$ are such that the integral exists in this form.

The argument of [3] is that the factor I given by

$$I = \left[\prod_{i<j=1}^{n} (z_i^{(2)} - z_j^{(2)})^2\right]\left[\prod_{i<j=0}^{n} (z_i^{(3)} - z_j^{(3)})^2\right]\left[\prod_{i=1}^{n}\prod_{j=0}^{n} (z_i^{(2)} - z_j^{(3)})\right] \tag{5.3.1}$$

is the dominant factor in the integrand for $p_2(z)$ and similarly for $p_3(z)$. For large n, the integral may be evaluated by expanding about the point in the space of integration where $|I|$ is largest, provided the arcs Γ_2, Γ_3 have been chosen so that this maximum is least.

In [3] a procedure was given for finding these arcs and the dominant factor in the asymptotic behavior of $p_2(z)$, $p_3(z)$. A Riemann surface $\mathscr{R}$ of three sheets is constructed by taking one copy of the complex plane cut along arcs joining $b_1 b_2$ and $b_3 b_4$. This is joined to another copy cut along arc $b_1 b_2$ and to one cut along $b_3 b_4$. The resulting surface is of genus zero and has four square-root branch points at $b_j, j = 1, \ldots, 4$, and an equation of the form (3.1.1) can be obtained to describe it in which $r(y, z)$ is of degree 3 in y and linear in z. (see [3, Appendix]. We call $\infty^{(1)}$ the point at ∞ on the sheet with both cuts.

Now for this surface the function $\phi(z)$ of Section 3.1 may be characterized as such that $\exp(\phi(z))$ is meromorphic with poles at $\infty^{(2)}$, $\infty^{(3)}$ and a double zero at $\infty^{(1)}$. The location of the cuts is chosen so that the discontinuity of $\operatorname{Re} \phi(z)$ across each cut is zero. If we define the meromorphic function $\psi(z)$ by

$$\psi(z) = \frac{d\phi}{dz} \tag{5.3.2}$$

then the real function $\rho_2(z)$, $z \in \Gamma_2$, is given from

$$\psi_+(z) - \psi_-(z) = 2\pi i \rho_2(z) \frac{|dz|}{dz}, \qquad z \in \Gamma_2, \tag{5.3.3}$$

and similarly for $\rho_3(z)$, $z \in \Gamma_3$.

In order for the above analysis to be applicable, it must turn out that $\rho_2, \rho_3 \geqslant 0$, on their respective arcs, which will not always be the case. If the densities are non-negative, let us label the sheets described above so that sheet a has cuts Γ_2, Γ_3, sheet b has cut Γ_2 and sheet c cut Γ_3. Then our analysis shows that the dominant factors in the asymptotic behavior of $p_2(z)$, $p_3(z)$ are

$$\begin{aligned} p_2(z) &\sim \exp(n\phi(z^{(b)})), \qquad z \notin \Gamma_2 \\ p_3(z) &\sim \exp(n\phi(z^{(c)})), \qquad z \notin \Gamma_3. \end{aligned} \tag{5.3.4}$$

We now consider how these results fit in with the conjecture of Section 3. We take $\mathscr{R}$ constructed above as the surface of the conjecture. In some cases it is found that sheet 1= sheet a and $S' = \Gamma_2 + \Gamma_3$. The curve S is a closed curve in $\mathbb{C}$ separating Γ_2 from Γ_3 and dividing $\mathbb{C}$ into $\mathbb{C} = C_2 + C_3 + S$. Let S divide sheet b into parts $\mathscr{R}_{b2}$, $\mathscr{R}_{b3}$, where $\mathscr{R}_{b2}$ abuts Γ_2, and similarly for sheet c, where $\mathscr{R}_{c3}$ abuts Γ_3. We have $\mathscr{R}_3 = \text{sheet } 3 = \mathscr{R}_{b3} + \mathscr{R}_{c2}$, and sheet 2 is what is left of $\mathscr{R}$.

In order that the predictions of Section 3.2 might apply, it is necessary that $f_2(z), f_3(z)$ be single valued, $z \in \mathscr{R}_0$. Equation (3.2.4) then gives

$$\begin{aligned} \chi_1(z^{(3)}) + f_2(z^{(1)})\chi_2(z^{(3)}) + f_3(z^{(1)})\chi_3(z^{(3)}) &= 0 \\ \chi_1(z^{(3)}) + f_2(z^{(2)})\chi_2(z^{(3)}) + f_3(z^{(1)})\chi_3(z^{(3)}) &= 0 \end{aligned} \qquad z \in C_2 \tag{5.3.5}$$

from which it follows that

$$\begin{aligned} \chi_2(z^{(3)}) &= 0 \\ \chi_1(z^{(3)}) &= -f_3(z^{(1)})\chi_3(z^{(3)}) \end{aligned} \qquad z \in C_2. \tag{5.3.6}$$

Similarly,

$$\begin{aligned} \chi_3(z^{(3)}) &= 0 \\ \chi_1(z^{(3)}) &= -f_2(z^{(1)})\chi_2(z^{(3)}) \end{aligned} \qquad z \in C_3. \tag{5.3.7}$$

For $z \in s$ with $z \in$ sheet c, (3.2.5) gives, with the help of (5.3.6),

$$(f_3(z^{(c)}) - f_3(z^{(1)}))\chi_3(z-) = R(z+), \qquad z \in s \text{ and } z \in \text{sheet } c \tag{5.3.8}$$

which shows that $R(z)$ has a meromorphic continuation in $\mathscr{R}_{c2}$ given by $(f_3(z^{(c)}) - f_3(z^{(1)}))\chi_3(z^{(c)})$. Similarly, there is a continuation of $R(z)$ into

$\mathscr{R}_{b3}$ given by $(f_2(z^{(b)}) - f_2(z^{(1)}))\chi_2(z^{(b)})$. The function $R(z)$ is therefore meromorphic, $z \in \mathscr{R}$, and must have the form $\exp(n\phi(z)) \times$ (a function with a small number of poles, zeros). The predictions of the conjecture are therefore in agreement with (5.3.4) if we interpret $\exp(\phi(z)) = 0$ whenever $z \notin \mathscr{R}_3$.

From (3.2.3) we may in this case derive the relation, $z \notin S'$,

$$p_1(z) + \sum_{j=2}^{3} f_j(z^{(1)})\, p_j(z) + (2\pi i)^{-1} \sum_{j=2}^{3} \int_{\Gamma_j} dt (f_{j+}(t^{(1)}) - f_{j-}(t^{(1)}))\, p_j(t)(t-z)^{-1} = 0 \quad (5.3.9)$$

which is a generalization of (1.2.5). With (5.3.4) and (5.3.9), it is easy to show that $p_1(z) \sim \chi_1(z^{(3)})$, where $\chi_1(z^{(3)})$ is given by (5.3.6), (5.3.7). The prediction is that almost all the zeros of $p_1(z)$ approach S as in the general case, but the zeros of $p_2(z)$, $p_3(z)$ approach Γ_2, Γ_3, respectively, not the general situation. The asymptotic density of zeros of $p_2(z)$ is $\rho_2(z)$ and likewise $p_3(z)$.

We have compared these predictions to the results of numerical computations, some of which were reported in [3]. Whatever we have calculated is consistent with the predictions, but additional calculations would be of interest. For example [3], the surface

$$\mathscr{R}: 1.8y^3 - (1.2z + 4.3)y^2 + 2.6zy + z = 0 \quad (5.3.10)$$

is of the type discussed here, with branch points at $z = b_j$, $j = 1, ..., 4$,

$$\begin{aligned} b_1 &\approx -1.9298 &&: y \approx -0.664 \\ b_2 &= 0 &&: y = 0 \\ b_3 &= 2 &&: y = 2 \\ b_4 &\approx 4.95 &&: y \approx 2.998. \end{aligned} \quad (5.3.11)$$

The points at infinity are given by

$$\begin{aligned} &\infty^{(1)}: y = \infty \\ &\infty^{(2)}: y = 2.5 \\ &\infty^{(3)}: y = -1/3 \end{aligned} \quad (5.3.12)$$

and we have

$$\exp(\phi(z)) = [(y - 2.5)(y + 1/3)]^{-1}. \quad (5.3.13)$$

The arcs Γ_2, Γ_3 are segments of the real axis.

In one example we chose

$$f_2(z) = (1 - b_1 x)^{1/2}$$
$$f_3(z) = [(1 - b_3 x)(1 - b_4 x)]^{1/2} \tag{5.3.14}$$

which functions have poles of first, second order in the local variable at $z = b_2$. An excellent fit to the numerical results is given by taking the previous formulae with

$$R(z) = y^{-2} \exp(n\phi(z)). \tag{5.3.15}$$

We note that the formula (5.3.8) appears to hold throughout sheet c, not just on $\mathscr{R}_{c2}$.

If in (5.3.14) $f_2(z)$ is replaced by

$$f_2(z) = (1 - b_1 x)^{-1/2} \tag{5.3.16}$$

then a less accurate fit to be numerical results is given by

$$R(z) = [y(y - y(b_1))]^{-1} \exp(n\phi(z)). \tag{5.3.17}$$

For both these choices, the functions $\{f_j(j)\}$ are single valued for $z \in \mathscr{R}_0$, although in each case poles and present in this region.

In these examples, then, we have strong support for the conjecture based on a surface $\mathscr{R}$ constructed as described above with branch points b_j, $j = 1,..., 4$. If, however, we take the functions (5.3.14) with a different choice for the points $\{b_j\}$, another situation can arise. It may turn out that the functions $\rho_2(z)$, $\rho_3(z)$ as constructed by the method described are found to be not >0, an unacceptable situation, since they are supposed to represent the asymptotic density of zeros in polynomials $p_2(z)$, $p_3(z)$. As explained in [3] we must look for another way of solving the equations obtained from the saddle-point method. It was suggested that we look for a solution in which one of the densities, say $\rho_3(z)$, is zero for part of the interval $b_3 b_4$. It was shown that this may be done by following the previous approach with b_3 replaced by b_3'. This point is determined by the condition that $\rho_3(b_3') = 0$, and we showed that, with $b_2 = 0$,

$$b_3' = \frac{(b_1 + b_4)^3}{9(b_1 + b_4)^2 - 27 b_1 b_4}. \tag{5.3.18}$$

The surface $\mathscr{R}$, with square root branch points at $z = b_1$, $b_2 = 0$, b_3' and b_4 is

$$(y - z)^{-1} - (y - b_1)^{-1} - (y - b_4)^{-1} - y^{-1} = 0. \tag{5.3.19}$$

For example, consider the choice

$$\begin{aligned} b_1 &= -1 \\ b_2 &= 0 \\ b_3 &= \tfrac{1}{2} \\ b_4 &= 10 \end{aligned} \tag{5.3.20}$$

which leads to $b_3' = 27/37$. We label sheets a, b, c as before. It is found that $\infty^{(b)}$, $\infty^{(c)}$ correspond to

$$\begin{aligned} \infty^{(b)}&: y = y_b \approx -0.5119 \\ \infty^{(c)}&: y = y_c \approx 6.5119 \end{aligned} \tag{5.3.21}$$

and that

$$\exp(\phi(z)) = [(y - y_b)(y - y_c)]^{-1}. \tag{5.3.22}$$

From this it follows that $\mathscr{R}_3 =$ sheet $3 =$ sheet b and that $S = \Gamma_2$, the segment of the real axis $b_1 b_2$. There is a simple closed curve $S_1' \in \mathbb{C}$ that surrounds Γ_2 and passes through $z = b_3'$. We find that $S' = S_1' + \Gamma_3$, where Γ_3 is the line segment $b_3' b_4$, and sheet $1 =$ (that part of sheet a outside S_1') + (that part of sheet c inside S_1').

The choice of $\mathscr{R}$ means that $f_3(z)$ has branch points on sheets 1, 2 at $z = b_3$, a point inside the curve S_1', but the functions $f_1(z), f_2(z)$ are single-valued, $z \in \mathscr{R}_0$. The situation comes under case 2 of the conjecture, Section 3.3 and we expect that

$$\chi_1(z^{(3)}) + f_2(z^{(1)})\,\chi_2(z^{(3)}) + f_3(z^{(1)}\pm)\,\chi_3(z^{(3)}) = 0 \tag{5.3.23}$$

where $f_3(z^{(1)}\pm)$ mean the values of $f_3(z^{(1)})$ on either side of a cut ending at b_3 on sheet 1. We deduce that on the cut

$$\chi_3(z^{(3)}) = 0 \tag{5.3.24}$$

which we extend to the whole of $\mathscr{R}_3$ by analytic continuation. This result is consistent with (5.3.4), because, in this case,

$$\operatorname{Re} \phi(z^{(b)}) > \operatorname{Re} \phi(z^{(c)}), \qquad z \notin \Gamma_2. \tag{5.3.25}$$

With (5.3.24) the other equations of the conjecture become unambiguous and reduce to (note $f_2(z^{(1)}) = f_2(z^{(2)})$)

$$\chi_1(z^{(3)}) = -f_2(z^{(1)})\,\chi_2(z^{(3)}), \qquad z \notin \Gamma_2 \tag{5.3.26}$$

and the boundary condition (3.2.5) rewritten with the help of (5.3.26),

$$(f_{2+}(z^{(2)})-f_{2-}(z^{(2)}))\chi_{2-}(z^{(3)})=R_{+}(z^{(2)}), \qquad z\in\Gamma_2 \tag{5.3.27}$$

and similarly with the limits reversed. Again, the choice of $f_2(z)$ shows that $R(z)$ is meromorphic, $z\in\mathscr{R}$. Thus the conjecture predicts

$$\begin{aligned} p_2(z) &\underset{n\to\infty}{\sim} \chi_2(z^{(3)})=-0.5f_2^{-1}(z^{(1)})\,R(z^{(3)}) \\ p_1(z) &\underset{n\to\infty}{\sim} \chi_1(z^{(3)})=0.5R(z^{(3)}) \end{aligned} \qquad z\notin\Gamma_2 \tag{5.3.28}$$

where

$$R(z)=[(y-y_b)(y-y_c)]^{-n}\,h(y) \tag{5.3.29}$$

with $h(y)$ a meromorphic function of low degree and y is given by (5.3.19). There is strong numerical support for this prediction with $h(y)$ independent of n, but we are not sure of its precise form. We note that (5.3.28) is consistent with (5.3.4) and (5.3.9).

It is also of interest to remark that the numerical results are consistent with

$$p_3(z) \underset{n\to\infty}{\sim} 0.5f_3^{-1}(z^{(1)})\,R(z^{(c)}) \tag{5.3.30}$$

away from the interval b_3b_4. Such a refinement of (5.3.24) and (5.3.4) is not as yet within the scope of the conjecture.

The saddle-point method may also be applied to type II polynomials for the same choices of functions $\{f_j(z)\}$. The polynomial $q_1(z)$ defined by (3.2.12) is a generalized orthogonal polynomial of type II as defined by (2.4.4) and a representation of the form (2.4.5) may be used. The dominant factor in this integral is similar to I of (5.3.1) and the saddle point argument will lead to the prediction.

$$q_1(z) \underset{n\to\infty}{\sim} \exp(2n(\phi(z^{(b)})+\phi(z^{(c)}))), \qquad z\notin\Gamma_2,\Gamma_3. \tag{5.3.31}$$

With a suitable choice of the arbitrary additive constant in $\phi(z)$, this may be written

$$q_1(z) \underset{n\to\infty}{\sim} \exp(-2n\phi(z^{(a)})), \qquad z\notin\Gamma_2,\Gamma_3 \tag{5.3.32}$$

which, in the case of the first example of this section, is

$$q_1(z) \underset{n\to\infty}{\sim} \exp(-2n\phi(z^{(1)})), \qquad z\notin S' \tag{5.3.33}$$

consistent with the conjecture, which predicts that most zeros of $q_1(z)$ will lie near S'.

When the branch points are all real and the discontinuities of the functions $f_j(z)$, $j = 2, 3$, are positive, we have a situation treated rigorously by Gončar and Rahmanov [17]. It may be shown that the asymptotic behavior predicted above follows from their work.

6. Possible Methods of Proof

This section contains some suggestions for proving the conjecture for particular classes of function $\{F_j(x)\}$. We expect the conjecture to apply to some functions not included in any of these classes.

6.1. *Extension of Bernstein–Szegö Method*

To extend the Bernstein–Szegö method [45] to the case $m > 2$ we need to find a suitable generalization of (4.4.5), which relates the exact orthogonal polynomial to the polynomial for an approximate weight. This can be done, as we illustrate here, for type I polynomials with the help of the reproducing kernels studied in Section 2.3, and a similar procedure no doubt exists for type II polynomials.

We use the notation and assumptions of Sections 2.1–2.3, so that $F_1(x) = 1$. We suppose that $\bar{F}_j(x)$, $j = 1,..., m$, with $\bar{F}_1(x) = 1$, are functions that will be used to approximate $\{F_j(x)\}$, and that $\{\bar{P}_j(\rho^{(l)}, x)\}$, $\{\bar{Q}_j(\mu^{(l)}, x)\}$ are the corresponding polynomials, with the same normality assumptions as in Section 2.1. An approximation to the reproducing kernel (2.3.1) is defined by (remember $f_j(z) = F_j(z^{-1})$ and similarly $\bar{f}_j(z)$)

$$\bar{K}_{jk}(z, t) = z^{n+1} \sum_{l=1}^{m} \bar{P}_j(\rho^{(l)}, z^{-1})\, \bar{Q}_1(\mu^{(l)}, t^{-1})\, t^{\nu} f_k(t)(t-z)^{-1}, \qquad j, k = 2,..., m. \tag{6.1.1}$$

Integrating round a contour in the z-plane large enough to include all sheet 1 singularities of $\{f_j(z)\}$ we find, with the help of (2.1.10),

$$\begin{aligned} &\sum_{k=2}^{m} \int dt\, \bar{K}_{jk}(z, t)\, P_k(\rho^{(1)}, t^{-1})\, t^n \\ &\quad = \sum_{l=1}^{m} \sum_{k=2}^{m} \int dt\, z^{n+1} \bar{P}_j(\rho^{(l)}, z^{-1})[\bar{Q}_1(\mu^{(l)}, t^{-1})\, t^{\nu-1} - \bar{Q}_1(\mu^{(l)}, z^{-1})\, z^{\nu-1}] \\ &\quad \times (t-z)^{-1} f_k(t)\, P_k(\rho^{(1)}, t^{-1})\, t^{n+1}. \end{aligned} \tag{6.1.2}$$

For $l \neq 1$ the expression $[\;](t-z)^{-1}$ is a polynomial in t of degree $\nu - 2$ and so the corresponding contribution to (6.1.2) is zero by virtue of (2.2.2). For $l=1$ the expression is the sum of a polynomial in t of degree $\nu-2$ and

$$\bar{Q}_1(\mu^{(1)})|_\nu\,(t^{-1}-z^{-1})(t-z)^{-1} \tag{6.1.3}$$

where $Q|_\nu$ means the coefficient of x^ν in $Q(x)$. We thus obtain

$$\begin{aligned}\sum_{k=2}^{m} &\int dt\,\bar{K}_{jk}(z,t)\,P_k(\rho^{(1)},t^{-1})\,t^n\\ &= -z^n\bar{P}_j(\rho^{(1)},z^{-1})\,\bar{Q}_1(\mu^{(1)})|_\nu \sum_{k=2}^{m}\int dt\,f_k(t)\,P_k(\rho^{(1)},t^{-1})\,t^n\\ &= 2\pi i z^n\bar{P}_j(\rho^{(1)},z^{-1})\,\bar{Q}_1(\mu^{(1)})|_\nu\,P_1(\rho^{(1)})|_{n+1}\end{aligned} \tag{6.1.4}$$

from (1.3.1). Note that the last two factors in (6.1.4) cannot be zero on account of (2.1.3).

In terms of the polynomials $p_j(z)$, $j=2,\dots,m$, defined by (2.2.1) our results may be written

$$p_j(z)=\int dt\,\bar{M}_j(z,t)\sum_{k=2}^{m}\bar{f}_k(t)\,p_k(t),\qquad j=2,\dots,m \tag{6.1.5}$$

because of the reproducing property, and

$$\mu\bar{p}_j(z)=\int dt\,\bar{M}_j(z,t)\sum_{k=2}^{m}f_k(t)\,p_k(t),\qquad j=2,\dots,m \tag{6.1.6}$$

which is (6.1.4) with $\bar{p}_j(z)$ defined in the same way as $p_j(z)$. We have used

$$\bar{M}_j(z,t)=(2\pi i)^{-1}z^{n+1}\sum_{l=1}^{m}\bar{P}_j(\rho^{(l)},z^{-1})\,\bar{Q}_1(\mu^{(l)},t^{-1})\,t^\nu(t-z)^{-1}. \tag{6.1.7}$$

Subtracting (6.1.6) from (6.1.5) would give a generalization of (4.4.5), an integral equation for the column $\{p_k(t)\}$, which reduces to (4.4.5) when $m=2$. However, this equation is not likely to be useful for obtaining asymptotics, and we now derive one that might be more suitable.

From (6.1.5) we have

$$\begin{aligned}\sum_{j=2}^{m}&\bar{F}_j(z)\,p_j(z)-\sum_{j=2}^{m}f_j(z)\,p_j(z)\\ &=\sum_{j=2}^{m}\int dt(\bar{f}_j(z)-f_j(z))\,\bar{M}_j(z,t)\sum_{k=2}^{m}\bar{f}_k(t)\,p_k(t).\end{aligned} \tag{6.1.8}$$

Now we multiply by $\sum_{i=2}^{m} \int dz\, \bar{f}_i(\bar{t})\, \bar{M}_i(\bar{t}, z)$ and use (6.1.5), (6.1.6) on the left-hand side to obtain

$$\sum_{i=2}^{m} \bar{f}_i(\bar{t})\, p_i(\bar{t}) - \mu \sum_{i=2}^{m} \bar{f}_i(\bar{t})\, \bar{p}_i(\bar{t}) = \sum_{i=2}^{m} \int dz\, \bar{f}_i(\bar{t})\, \bar{M}_i(\bar{t}, z) \int dt \sum_{j=2}^{m} (\bar{f}_j(z) - f_j(z))\, \bar{M}_j(z, t) \sum_{k=2}^{m} \bar{f}_k(t)\, p_k(t). \tag{6.1.9}$$

This is an integral equation for the single quantity

$$u(z) = \sum_{j=2}^{m} \bar{f}_j(z)\, p_j(z). \tag{6.1.10}$$

From its solution we can obtain $p_j(z), j = 2, \ldots, m$ (i.e. $P_j(\rho^{(1)}, x)$) from (6.1.5) and the polynomial $P_1(\rho^{(1)}, x)$ from

$$z^n P_1(\rho^{(1)}, z^{-1}) = -(2\pi i)^{-1} \int dt \sum_{j=2}^{m} f_j(t)\, p_j(t)(t-z)^{-1} \tag{6.1.11}$$

with z inside the contour of integration, an equation deduced from (1.3.1).

For (6.1.9) to be useful, it is necessary that its kernel be small, and this will only be the case if the integration contour is chosen appropriately. Let us suppose that S' (see Section 3.1) is such that its complement is connected, which corresponds to the situation in which sheet 1 of $\mathscr{R}$ is connected. We collapse the integration contour onto S' and also take the difference of the two versions of (6.1.9) obtained by taking the limit as $\bar{t} \to S'$ from opposite sides, giving

$$v(\bar{t}) - \mu \bar{v}(\bar{t}) = \int_{S'} dz \int_{S'} dt\, \bar{H}(\bar{t}, z)(\bar{H}(z, t) - H(z, t))\, v(t), \; \bar{t} \in S', \tag{6.1.12}$$

where

$$v(z) = \sum_{j=2}^{m} \omega_j(z)\, p_j(z) \tag{6.1.13}$$

$$\bar{v}(z) = \sum_{j=2}^{m} \bar{\omega}_j(z)\, \bar{p}_j(z) \tag{6.1.14}$$

$$H(z, t) = \sum_{j=2}^{m} \omega_j(z)\, \bar{M}_j(z, t) \tag{6.1.15}$$

$$\bar{H}(z, t) = \sum_{j=2}^{m} \bar{\omega}_j(z)\, \bar{M}_j(z, t) \tag{6.1.16}$$

with $\omega_j(z)$, $\bar{\omega}_j(z)$ being the discontinuities of $f_j(z), \bar{f}_j(z)$ on sheet 1 across S'.

Now the arbitrary additive constant in $\phi(z)$ of Section 3.1 may be chosen so that

$$\operatorname{Re} \sum_{j=1}^{m} \phi(z^{(j)}) = 0 \tag{6.1.17}$$

for the left-hand side is certainly constant, being a bounded harmonic function, $z \in \mathbb{C}$.

Let us suppose that we are considering case 1 of the conjecture, Section 3.2. We take the functions $\{\bar{f}_j(z)\}$ to be meromorphic, $z \in \mathscr{R}$, with poles restricted to $\mathscr{R}_m$. If these functions were independent of n, we expect that, as in Section 4.1,

$$\sum_{j=1}^{m} \bar{f}_j(z)\, \bar{p}_j(z) = \exp(n\phi(z))\, O(1) \tag{6.1.18}$$

$$\bar{p}_j(z) = \exp(n\phi^{(m)}))\, O(1) \tag{6.1.19}$$

and, with normalizations chosen to satisfy (2.1.9),

$$z^{\nu} Q_1(\mu^{(l)}, z^{-1}) = \exp(-n\phi(z^{(1)}))\, O(1). \tag{6.1.20}$$

In fact the functions $\{\bar{f}_j(z)\}$ will depend on n, being chosen to approximate $\{f_j(z)\}$ with increasing accuracy as n increases, but we assume that this is done so that (6.1.18)–(6.1.20) still hold.

We deduce that

$$\exp(-n \operatorname{Re} \phi(z^{(1)}))\, \bar{v}(z) = O(1), \qquad z \in S', \tag{6.1.21}$$

and we expect a similar relation to hold for $v(z)$. It is sensible to rewrite (6.1.12) as an equation for $V(z)$,

$$V(z) = \exp(-n \operatorname{Re} \phi(z^{(1)}))\, v(z), \tag{6.1.22}$$

giving

$$V(\bar{t}) - \mu \bar{V}(\bar{t}) = \int_{S'} dz \int_{S'} dt\, \bar{G}(\bar{t}, z)(\bar{G}(z, t) - G(z, t))\, V(t), \qquad \bar{t} \in S', \tag{6.1.23}$$

where

$$\bar{V}(z) = \exp(-n \operatorname{Re} \phi(z^{(1)}))\, \bar{v}(z), \qquad z \in S' \tag{6.1.24}$$

$$G(z, t) = \exp(-n \operatorname{Re} \phi(z^{(1)}))\, H(z, t) \exp(n \operatorname{Re} \phi(t^{(1)})), \qquad z, t \in S' \tag{6.1.25}$$

$$\bar{G}(z, t) = \exp(-n \operatorname{Re} \phi(z^{(1)}))\, \bar{H}(z, t)\, \exp(n \operatorname{Re} \phi(t^{(1)})), \qquad z, t \in S'. \tag{6.1.26}$$

We see that $\bar{G}(\bar{t}, z) = O(1)$ except perhaps near $z = \bar{t}$, and that

$$\bar{G}(z, t) - G(z, t) = \sum_{j=2}^{m} (\bar{\omega}_j(z) - \omega_j(z)) \exp(-n \operatorname{Re} \phi(z^{(1)})) \times \bar{M}_j(z, t) \exp(n \operatorname{Re} \phi(t^{(1)})). \tag{6.1.27}$$

Now

$$\exp(-n \operatorname{Re} \phi(z^{(1)}) \bar{M}_j(z, t) \exp(n \operatorname{Re} \phi(t^{(1)})) = \exp(n(\operatorname{Re} \phi(z^{(m)}) - \operatorname{Re} \phi(z^{(1)}))) O(1), \qquad z, t \in S'. \tag{6.1.28}$$

Thus for the method to work, it must be shown that it is possible to choose $\{\bar{f}_j(z)\}$ so that (6.1.18)–(6.1.20) hold and at the same time

$$|\bar{\omega}_j(z) - \omega_j(z)| \exp(n(\operatorname{Re} \phi(z^{(m)}) - \operatorname{Re} \phi(z^{(1)}))) \to 0, \qquad z \in S' \tag{6.1.29}$$

the approach to zero being adequately fast.

Given the necessary results from approximation theory on Riemann surfaces, the proof of the asymptotic conjecture for appropriate functions $\{F_j(x)\}$ should follow as in [37, 31]. We expect that the above equations may be modified to deal with the case when the complement of S' is not connected, if this situation is possible.

6.2. *Alternative Integral Equation Method*

The method of deriving asymptotics using the Bernstein–Szegö integral equation [45] requires the discontinuity $\omega(z)$ to have a particular form, namely (1.1.2), where $\sigma(z)$ is strictly positive, later generalized to non-zero [37]. This is because the polynomial $p(z)$ is related through the integral equation to the polynomial for weight corresponding to $\sigma(z)$ of the form (polynomial in $z)^{-1}$, or, in other words, the polynomial corresponding to $\{f_j(z)\}$ meromorphic on the surface $y^2 = 1 - z^2$ with poles restricted to the second sheet. The analogous procedure was used in [31] and has been suggested in Section 6.1 for $m > 2$. If the dominant singularity of $\omega(z)$ at $z = \pm 1$ is not inverse square root, then the Bernstein–Szegö integral equation method will fail, because the approximate weight cannot be made close enough to the exact weight.

We now outline an approach to the proof of the asymptotic conjecture in the case $m = 2$ for functions having dominantly power law singularities at the branch points of $\mathscr{R}$. We suppose that $f_1(z) = 1$ and $f_2(z)$ has the form (4.4.2), with S as described in connection with this equation. Near each end b_j of S we require

$$\omega(z) = X_+^{-1/2}(z)\, \sigma(z) = (z - b_j)^{\nu_j} \rho_j(z), \qquad z \in S \tag{6.2.1}$$

where $\rho_j(z)$ is a smooth, non-vanishing function near $z = b_j$. Elsewhere, $\omega(z)$ is to be adequately smooth.

The method again involves the construction of an approximation $\bar{K}(z, t)$ to the reproducing kernel written in the form

$$\bar{K}(z, t) = (\bar{p}(z)\bar{p}^*(t) - \bar{p}^*(z)\bar{p}(t))(t-z)^{-1}\,\omega(t). \tag{6.2.2}$$

The polynomials $\bar{p}(z)$, $\bar{p}^*(z)$ must be chosen to approximate the orthogonal polynomial $p_2(z)$ corresponding to degree n, $n+1$, respectively.

Now, since $\bar{K}(z, t)$ is a polynomial in t of degree n, with coefficient of t^n proportional to $\bar{p}(z)$, we have

$$\int_S dt\,\bar{K}(z, t)\,p(t) = \lambda_n \bar{p}(z) \tag{6.2.3}$$

where we have written $p(z)$ for $p_2(z)$ of degree n. In addition, if we have chosen $\bar{p}(z)$, $\bar{p}^*(z)$ wisely, it should be possible to show that

$$\int_S dt\,\bar{K}(z, t)\,p(t) = p(t) + \int_S dt\,L(z, t)\,p(t) \tag{6.2.4}$$

where $L(z, t)$ is small in an appropriate sense for large n. Together (6.2.3), (6.2.4) give the integral equation

$$p(z) = \lambda_n \bar{p}(z) - \int_S dt\,L(z, t)\,p(t) \tag{6.2.5}$$

which can be solved by iteration for large n.

We base the construction of $\bar{p}(z)$, $\bar{p}^*(z)$ on the asymptotic behavior (3.2.7),

$$p(z) \sim \chi_+(z) + \chi_-(z), \tag{6.2.6}$$

valid for $z \in S$ except near to each end b_j. (We have written $\chi(z)$ for $\chi_2(z)$). Near an end we find

$$\chi(z) \sim C_1(z - b_j)^{1/2v_j - 1/4} \exp(n(\phi(z) - \phi(b_j))). \tag{6.2.7}$$

From the results for Jacobi polynomials [45], which could be extended by the use of Christoffel's formula (Section 2.5), and their generalization (Section 5.1), we expect that, near an end b_j

$$p(z) \sim C_2(z - b_j)^{-1/2v_j} J_{v_j}(-in(\phi(z) - \phi(b_j))). \tag{6.2.8}$$

The n-dependent constants C_1, C_2 are related by the requirement that the expansion of (6.2.8) for large argument of the Bessel function coincides with (6.2.6) after substituting (6.2.7). This leads to

$$C_1 = C_2(2\pi\lambda_j n)^{-1/2} \exp(-\tfrac{1}{2}i\pi v_j) \tag{6.2.9}$$

where

$$\phi(z) - \phi(b_j) = \lambda_j(z - b_j)^{1/2}, \qquad z \approx b_j. \tag{6.2.10}$$

Note that the region of validity of (6.2.8) will decrease with increasing n.

We suggest that $\bar{p}(z)$ be chosen as a polynomial that approximates the expressions (6.2.6), (6.2.8) on the corresponding parts of S, and similarly $\bar{p}^*(z)$. It should be possible to do this with small error.

It may be that a zero of $Y(z)$ of (3.7.7) lies on S, so that three arcs of S meet there. The form (6.2.6) fails near such a point, but an appropriate expression to use can be obtained from an asymptotic analysis of the differential equation (5.1.16) as described by Olver [38].

6.3. *Other Approaches*

Since no general proof of the conjecture exists at present, it is of interest to consider restricted classes of functions $\{F_j(x)\}$. Riemann modules [12, 13] give rise to one such class. In Sections 4.7, 5.1 we have treated cases in which each function $F_j(x)$, $j = 1,..., m$, corresponds to the first component of an element in a module. The examples of meromorphic functions and of Sections 4.6, 5.2 also correspond to modules, although we have not used this property.

For functions corresponding to a module, it may be shown that the H–P polynomials (type I) and remainder function satisfy differential equations with polynomial coefficients of degree independent of n. With the help of the dual module, similar results can be demonstrated for type II polynomials. Not all the coefficients in the differential equations can be evaluated immediately. However, in the simple example of Section 5.1, where $m = 2$, we have shown that it might be possible to deduce their asymptotic behavior with the help of the Liouville–Green method [38], and so obtain H–P polynomial asymptotics. If error bounds in this method could be calculated for the case of higher order differential equations, then there is a chance that our approach could be extended to $m > 2$.

The multiple integral formulae of Section 2.4 have proved useful (see Section 5.3) in obtaining predictions by way of heuristic arguments involving the saddle-point method [8]. There is a possibility, perhaps not very large, that this method could be made rigorous. More study may be warranted.

APPENDIX 1

Here we demonstrate some properties of matrices needed in the discussion of Section 4.7 and note some relations among generalized hypergeometric functions that are a consequence of our approach. In this appendix, invariant will be taken to mean invariant if any integers are added to the parameters $a_1 \cdots a_m c_2 \cdots c_m$.

From (4.7.9) we find that

$$(TG^{-1})_{jl}\,(GT^{-1})_{lk} = \frac{L(1-a_j)}{L(1-a_k)} \left(\frac{\prod_{t\neq j} L(a_t - a_j)}{\prod_t L(c_t - a_j)} \right) \left(\frac{\prod_{t\neq l} L(c_l - c_t)}{\prod_{t\neq j,k} L(c_l - a_t)} \right), \quad j \neq k \qquad \text{(A1.1)}$$

and

$$(TG^{-1})_{jl}\,(GT^{-1})_{lj} = \left(\prod_{t\neq j} \frac{L(a_t - a_j)}{L(c_l - a_t)} \right) \left(\prod_{t\neq l} \frac{L(c_l - c_t)}{L(c_t - a_j)} \right) \qquad \text{(A1.2)}$$

where

$$L(x) = \Gamma(x)\,\Gamma(1-x) = \frac{\pi}{\sin \pi x}. \qquad \text{(A1.3)}$$

Since

$$V_0 = KJK^{-1} \qquad \text{(A1.4)}$$

and the elements of K, C^2 are invariant, the form (4.7.21) of J shows that V_0 is invariant. Moreover, it is clear that (4.7.22) holds with D_{jj} given by

$$D_{jj} = \frac{1}{[L(1-a_j)]^2} \cdot \frac{\prod_t L(c_t - a_j)}{\prod_{t\neq j} L(a_t - a_j)}. \qquad \text{(A1.5)}$$

Now suppose we have two contiguous sets of parameters and denote the corresponding $\underline{W}$ by $\underline{W}(1)$, $\underline{W}(2)$, etc. Then from (4.7.18), (4.7.28) and (4.7.24) we have

$$\begin{aligned} \underline{W}(1)^T \,\underline{\tilde{W}}(2) &= \underline{Y}^{(\infty)}(1)^T A T(1)^T K^T (K^T)^{-1} D \bar{K}^{-1} \bar{K} \bar{T}(2) \bar{A} \underline{\bar{Y}}^{(\infty)} \\ &= \underline{Y}^{(\infty)}(1)^T\, T(1)^T D \bar{T}(2)\, \underline{Y}^{(\infty)}(2) \end{aligned} \qquad \text{(A1.6)}$$

or

$$\underline{W}(1)^T \,\underline{\tilde{W}}(2) = \sum_{k=1}^{m} y_k^{(\infty)}(1;x)\, \bar{y}_k^{(\infty)}(2;x)\, G_{1k}(1)\, \bar{G}_{1k}(2)\, D_{kk}. \qquad \text{(A1.7)}$$

There is a similar relation involving $y_k^{(0)}$, for which we sketch the derivation. Using (4.7.13), (4.7.28) and (4.7.24) we find

$$\underline{W}(1)^T \underline{\tilde{W}}(2) = \underline{Y}^{(0)}(1)^T C(G(1)^T)^{-1} T(1) D\bar{T}(2) \bar{G}(2)^{-1} \bar{C}\underline{\bar{Y}}^{(0)}(2). \quad \text{(A1.8)}$$

Now (4.7.22) is equivalent to

$$(\bar{G}^T)^{-1} \bar{T}DTG^{-1}C^2 = C^2(\bar{G}^T)^{-1} \bar{T}DTG^{-1} \quad \text{(A1.9)}$$

and, since C^2 is diagonal with no two diagonal elements the same, it must be that

$$(\bar{G}^T)^{-1} \bar{T}DTG^{-1} = Q \quad \text{(A1.10)}$$

with Q diagonal. The elements of Q may be found by equating diagonal elements in

$$\bar{T}DTG^{-1} = \bar{G}^T Q. \quad \text{(A1.11)}$$

In addition it may be seen from the form (4.7.9) that a diagonal matrix M exists so that

$$T(1)\, G(1)^{-1} M(1)^{-1} = T(2)\, G(2)^{-1} M(2)^{-1}. \quad \text{(A1.12)}$$

In fact

$$M_{jj} = \frac{1}{\Gamma(c_j - a_j)} \left(\prod_t \frac{\Gamma(c_t)}{\Gamma(a_t)} \right) \left(\prod_{t \neq j} \frac{\Gamma(c_j - c_t)}{\Gamma(c_j - a_t)} \right). \quad \text{(A1.13)}$$

Thus (A1.8) becomes

$$\underline{W}(1)^T \underline{\tilde{W}}(2) = \underline{Y}^{(0)}(1)^T M(1)\, M(2)^{-1}\, Q(2)\, \underline{\bar{Y}}^{(0)}(2) \quad \text{(A1.14)}$$

and if use is made of the form of Q it may be shown that

$$\underline{W}(1)^T \underline{\tilde{W}}(2) = \sum_{k=1}^{m} y_k^{(0)}(1; x)\, \bar{y}_k^{(0)}(2; x)\, \mu_k(1, 2) \quad \text{(A1.15)}$$

where

$$\mu_k(1, 2) = M_{kk}(1)\, \bar{B}_k \bar{M}_{kk}(2) \quad \text{(A1.16)}$$

with

$$B_k = L(c_k - a_k) \left(\prod_{t \neq k} \frac{L(c_k - a_t)}{L(c_k - c_t)} \right). \quad \text{(A1.17)}$$

Now since $\underline{W}(1)^T \tilde{\underline{W}}(2)$ is unchanged after continuation round any branch point, it must be a rational function of x with poles possible only at $x = 0, 1, \infty$. The order of the poles at $x = 0, \infty$ is immediately clear from (A1.15), (A1.7), while at $x = 1$ the question is dealt with by applying the transformation P of (4.7.36) and its dual $(P^T)^{-1}$ to $\tilde{W}$. These ideas are used to construct the type II polynomials in Section 4.7.2.

We note the form of (A1.15) in the special case when the two sets of parameters are the same. In this case the rational function $\underline{W}(1)^T \tilde{\underline{W}}(1)$ is a constant and we have

$$\sum_{k=1}^{m} y_k^{(0)}(x)\, \bar{y}_k^{(0)}(x)\, \mu_k(1, 1) = \mu_1(1, 1). \tag{A1.18}$$

In particular, with $m = 2$,

$$\begin{aligned} &c_2^2(c_2+1)(c_2-1)\, {}_2F_1(a_1, a_2; c_2; x)\, {}_2F_1(-a_1, -a_2; -c_2; x) \\ &\quad - a_1 a_2 (c_2 - a_1)(c_2 - a_2)\, x^2\, {}_2F_1(1 + a_1 - c_2, 1 + a_2 - c_2, 2 - c_2; x) \\ &\quad \times {}_2F_1(1 - a_1 + c_2, 1 - a_2 + c_2; 2 + c_2; x) = c_2^2(c_2+1)(c_2-1). \end{aligned} \tag{A1.19}$$

Other relations follow from (A1.7).

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